One point statistics of the induced electric field in quasi-normal magnetofluid turbulence

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Abstract

We study one point statistical properties of the induced turbulent electric field for a magnetohydrodynamic (MHD) plasma under the quasi-normal approximation. Assuming exact Gaussianity for both the velocity field and the magnetic field, and different degrees of correlations between their Cartesian components, we derive the probability distribution function (PDF) for the Cartesian components of the electric field, e_i. We show that the PDF reduces in some canonical cases to an exponential function of the form $exp(-|e_i|)$. To study deviations from these results in the more realistic case in which the velocity and magnetic fields are not exactly normal but quasi-normal instead, we perform three dimensional numerical simulations of the MHD equations at moderate Reynolds numbers. For turbulent relaxation from an initial condition, we find that the analytical results give a very good first order approximation to the computed PDF.

I. INTRODUCTION

The induced electric field plays a key role in the slow evolution of a plasma. Such an evolution is usually described by the magnetohydrodynamic (MHD) equations. In these equations, the electric field $\mathbf{E}$ is eliminated in terms of the magnetic field $\mathbf{B}$ by means of a simplified Ohm's Law:

$$\mathbf{E} = -\mathbf{V} \times \mathbf{B} + \eta \mathbf{j} \tag{1}$$

where η is the resistivity, $\mathbf{j} = \nabla \times \mathbf{B}$ is the electric current and $\mathbf{V}$ is the velocity field. The first term in the r.h.s. of Eq. (1) is referred to as the induced electric field. Ohm's law allows to write the MHD induction equation:

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{V} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}, \tag{2}$$

which states that the time evolution of the magnetic field is governed by the curl of the induced electric field.

Here we study the statistical properties of the induced electric field in fully developed MHD turbulence. In this scenario, both the velocity and the magnetic field are usually modeled as homogeneous, isotropic random variables. Nonetheless, in more realistic situations the dynamical fields need to be decomposed as:

$$\mathbf{V} = \mathbf{V}_0 + \mathbf{v} \tag{3}$$

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{b} \tag{4}$$

where $\mathbf{V}_0 = \langle \mathbf{V} \rangle$, $\mathbf{B}_0 = \langle \mathbf{B} \rangle$ and the $\langle \; \rangle$ operator denotes an ensemble average. Note that, by definition,

$$\langle \mathbf{v} \rangle = 0 = \langle \mathbf{b} \rangle \tag{5}$$

It is important to note that the statistical behavior of $\mathbf{v}$ and $\mathbf{b}$ exerts an influence on the evolution of the mean fields. For instance, taking the ensemble average of Eq. (2) we obtain:

$$\partial_t \mathbf{B_0} = \nabla \times (\mathbf{V_0} \times \mathbf{B_0}) + \nabla \times \langle \mathbf{v} \times \mathbf{b} \rangle + \eta \nabla^2 \mathbf{B_0}, \tag{6}$$

The second term in the r.h.s. of Eq. (6) reflects the action of the mean induced electric field of the fluctuations on the mean magnetic field. This term is responsible for the generation of large scale magnetic fields from a turbulent velocity field, a phenomenon known as the "dynamo effect" (see for instance [1]). It is this fluctuating component of the electric field,

$$\mathbf{e} = -\mathbf{v} \times \mathbf{b}, \tag{7}$$

that we are interested in in this paper. In principle, all the one point statistical properties of $\mathbf{e}$ can be derived from its Probability Distribution Function (PDF). The PDF for $\mathbf{e}$, in turn, may be computed in terms of the joint PDFs for $\mathbf{v}$ and $\mathbf{b}$. As a natural starting point, we assume that both $\mathbf{v}$ and $\mathbf{b}$ are Gaussian random variables.

It has been long known that the velocity field in fluid turbulence may be regarded as a quasi-normal, random (vector) variable. Early experiments (see for example Ref. [2]) demonstrated that the PDFs for the components of the velocity field are very close to Gaussian distributions, and that the Kurtosis ($K(x) = \langle x^4 \rangle / \langle x^2 \rangle^2$, where $\langle x \rangle = 0$) of these components is accordingly very close to the Gaussian value of 3. This result is also valid in MHD, where both $\mathbf{v}$ and $\mathbf{b}$ are approximately Gaussian (see for instance [3]). There is also substantial direct observational evidence that the magnetic field fluctuations in the solar wind are normal to a good approximation [4,5] It is worth mentioning that many closure theories for turbulence, such as the EDQNM model (see for instance [3] for a review) rely on the quasi-normal approximation [6,7], which consists of the assumption that forth order moments of the fields $\mathbf{v}$ and $\mathbf{b}$ are exactly Gaussian, which closes the infinite hierarchy of statistical moments. Here, instead, we simply assume exact Gaussianity of these fields for the sole purpose of obtaining an approximate PDF for the electric field. How far or how close the measured or observed electric field is from the approximate analytical result we find, is a matter of further work. Our preliminary numerical results seem to show that at least for moderate Reynolds numbers the approximation is fairly good.

II. ANALYTICAL MODEL FOR THE PDF OF THE INDUCED ELECTRIC FIELD

We want to obtain an expression for the PDF of any component of $\mathbf{e}$, say

$$e_l = v_i b_j - v_j b_i \tag{8}$$

where the cyclic indices $\{l, i, j\}$ all take different values between 1 and 3, according to $\mathbf{e} = -\mathbf{v} \times \mathbf{b}$. For simplicity in the notation, we will consider a random variable z of the form:

$$z \equiv x_1 x_2 - x_3 x_4, \tag{9}$$

which simply reflects the change in variables: $\{e_l, v_i, b_j, v_j, b_i\} \rightarrow \{z, x_1, x_2, x_3, x_4\}$. An important simplification comes from the hypothesis of variance isotropy [8](that we will use throughout the paper, unless explicitly stated). Variance isotropy, that is, $\sigma_{v_i} = \sigma_{v_j}$ and $\sigma_{b_i} = \sigma_{b_j}$ for all $i, j = 1, 2, 3$ implies $\sigma_{v_i}\sigma_{b_j} = \sigma_{v_j}\sigma_{b_i}$, or in terms of Eq. (9),

$$\sigma_1\sigma_2 = \sigma_3\sigma_4 \tag{10}$$

where $\sigma_x = \sigma(x)$ stands for the standard deviation of x and $\sigma_i \equiv \sigma(x_i)$.

A. Uncorrelated Gaussians

Calculations are much simplified under the hypothesis of statistical independence of the variables x_1, x_2, x_3, x_4. A first point to note is that if x_1 and x_2 are independent, then:

$$\langle (x_1 x_2)^n \rangle = \langle x_1^n \rangle \langle x_2^n \rangle \tag{11}$$

This expression in turn simplifies the calculation of the moments of combinations of the various $\{x_i\}$. In particular:

$$\sigma(x_1 x_2) = \sigma_1 \sigma_2, \tag{12}$$

$$K(x_1 x_2) = K_1 K_2 = 9, \tag{13}$$

$$\sigma(x_1 x_2 - x_3 x_4) = \sqrt{2}\sigma_1\sigma_2, \tag{14}$$

$$K(x_1 x_2 - x_3 x_4) = \frac{1}{2} K_1 K_2 + \frac{3}{2} = 6, \tag{15}$$

where $K_x = K(x)$ is the Kurtosis of x and $K_i \equiv K(x_i)$. The numerical values given for the Kurtosis only hold when the variables $\{x_i\}$ are all Gaussians, in which case $K_i = 3$ for all i.

We now compute the PDF for a component of the electric field. We proceed in two steps: we first compute the PDF for a variable of the form:

$$s \equiv x_1 x_2, \tag{16}$$

and we then proceed to compute the distribution of:

$$z \equiv s_1 - s_2, \tag{17}$$

where s_1 and s_2 are both of the form Eq. (16). We use Eq. (A3) from Appendix A:

$$f(s) = \int f_1(x_1) f_2(x_2) \delta(s - x_1 x_2) dx_1 dx_2 = \int f_1(x_1) f_2(s/x_1) \frac{dx_1}{|x_1|} \tag{18}$$

where $f(x)$ stands for the PDF of the random variable x, $f_i \equiv f(x_i)$ and we used the relation $\delta(s - x_1 x_2) = |x_1|^{-1} \delta(x_2 - s/x_1)$. Note that in Eq. (18) we made use of the statistical independence to write $f(x_1, x_2) = f_1(x_1) f_2(x_2)$. We now replace f_1 and f_2 in the integral by Gaussians of dispersions σ_1 and σ_2:

$$f_i \equiv \frac{1}{\sqrt{2\pi}\sigma_i} exp\left[-\frac{1}{2}\left(\frac{x_i}{\sigma_i}\right)^2\right] \tag{19}$$

Assuming $s > 0$, we change variables $x_1 \to u \equiv x_1/(\sigma_1\sqrt{2s})$, so that the integral Eq. (18) reads:

$$f(s) = \frac{1}{\pi\sigma_s} \int_0^\infty exp\left[-s\left(u^2 + \frac{1}{4\sigma_s^2 u^2}\right)\right] \frac{du}{u} = \frac{1}{\pi\sigma_s} K_0\left(\frac{s}{\sigma_s}\right), \qquad s > 0, \tag{20}$$

where $\sigma_s \equiv \sigma_1\sigma_2$ (see Eq. (12)) and $K_0(x)$ is a modified Bessel function of the second kind. It is evident that $f(s)$ is an even function of s ($f(s) = f(-s)$), so we can replace $s \to |s|$ to obtain:

$$f(s) = \frac{1}{\pi\sigma_s} K_0\left(\frac{|s|}{\sigma_s}\right). \tag{21}$$

We can now compute moments of arbitrary (even) order (odd moments vanish):

$$\langle s^n \rangle = \frac{1}{\pi \sigma_s} \int s^n K_0(\frac{|s|}{\sigma_s})\ ds = \sigma_s^n [(n-1)!!]^2 \tag{22}$$

As a consistency check, we note that the same result can be obtained from Eq. (11) and the well known result for even moments of a Gaussian variable. Also, Eq. (22) implies $K_s = 9$, in agreement with Eq. (13).

Now we are in a position of finding the PDF for the components of the induced electric field. We consider a variable of the form (see Eq. (17)) $z = s_1 - s_2$, where $f_i(s_i) = (1/\pi\sigma_{s_i})K_0(|s_i|/\sigma_{s_i})$. In this case, Eq. (A3) yields:

$$f(z) = \int f_1(s_1) f_2(s_2) \delta(z - s_1 + s_2)\ ds_1\ ds_2 = \int f_1(s_1) f_2(z - s_1) ds_1 \tag{23}$$

where we used the parity of f_2. We immediately notice that the problem reduces to the convolution of two K_0 functions. The convolution theorem for cosine Fourier transforms for even functions f_1 and f_2, reads:

$$\frac{1}{2} \int f_1(s) f_2(z-s) ds = \sqrt{\frac{\pi}{2}} F^{-1}[F(f_1)F(f_2)], \tag{24}$$

where F stands for the cosine Fourier transform. The direct transforms of Eq. (21) are straightforward. Assuming $z > 0$,

$$F(f_i) = \frac{1}{\sqrt{2\pi}\sqrt{1 + (k\sigma_{s_i})^2}}, \qquad i = 1, 2. \tag{25}$$

If $\sigma_{s_1} = \sigma_{s_2}$ (which is ensured by the variance isotropy condition, Eq. (10)), we can compute the inverse transform in Eqs.(23)-(24):

$$f(z) = \frac{1}{2\sigma_{s_1}} exp\left(-\frac{z}{\sigma_{s_1}}\right), \qquad z > 0. \tag{26}$$

But we know that $f(z) = f(-z)$, so we can write in general (see Eqs. (8),(9))

$$f(e_l) = \frac{1}{\sqrt{2}\sigma_{e_l}} exp\left(-\frac{\sqrt{2}}{\sigma_{e_l}}|e_l|\right), \tag{27}$$

where, according to Eq. (14),

$$\sigma_{e_l} = \sqrt{2}\sigma_{v_i}\sigma_{v_j}, \qquad i \neq j. \tag{28}$$

It can be checked by simple integration from Eq. (27) that $f(e_l)$ is properly normalized, and that the standard deviation of e_l is precisely σ_{e_l}.

B. Dynamo-type correlated Gaussians

There are at least two cases in which the hypothesis of statistical independence of the variables x_1, x_2, x_3, x_4 is inconsistent with the physical situation: (a) in the presence of cross helicity $H_c \equiv \langle \mathbf{v} \cdot \mathbf{b} \rangle$ and (b) in the presence of an efficient dynamo. This inconsistency is evident in the former case, since statistical independence implies $H_c = \langle v_i b_i \rangle = \langle v_i \rangle \langle b_i \rangle = 0$ (where we used Eq. (5) and the standard repeated index summation notation). In the latter case, the most relevant quantity is the mean induced electric field of the fluctuations, which is precisely $\langle \mathbf{e} \rangle = -\langle \mathbf{v} \times \mathbf{b} \rangle$. If the components of $\mathbf{v}$ and $\mathbf{b}$ are statistically independent, then $\langle \mathbf{e} \rangle = 0$. It is important to note that the converse is not true. For instance, $\langle e_z \rangle = 0$ implies $\langle v_x b_y \rangle = \langle v_y b_x \rangle$, but this does not imply statistical independence.

We study the case $H_c = 0$, $\epsilon \neq 0$ (dynamo type) in this section, and leave the case $H_c \neq 0$, $\epsilon = 0$ (cross helicity type) for the next section. The calculations we are going to perform are similar to the ones in the previous section. We will assume that the pairs (x_1, x_2) and (x_3, x_4) are each well described by Gaussian joint PDFs f_{12} and f_{34} of the form [9]:

$$f_{ij} \equiv \frac{1}{2\pi\sigma_i\sigma_j\sqrt{1-\rho_{ij}^2}} exp\left\{-\frac{1}{2(1-\rho_{ij}^2)}\left[\left(\frac{x_i}{\sigma_i}\right)^2 - 2\rho_{ij}\frac{x_i}{\sigma_i}\frac{x_j}{\sigma_j} + \left(\frac{x_j}{\sigma_j}\right)^2\right]\right\} \tag{29}$$

where

$$\rho_{ij} \equiv \frac{\langle x_i x_j \rangle}{\sigma_i \sigma_j} \tag{30}$$

Note that the mean value of z can be written in terms of ρ_{12} and ρ_{34}:

$$\langle z \rangle = (\rho_{12} - \rho_{34})\sigma_1\sigma_2 \tag{31}$$

That is, the mean induced electric field is non-zero, unless $\rho_{12} = \rho_{34}$. We now proceed as in the previous section. We first compute the PDF for $s = x_1 x_2$ from Eq. (A3). The calculation is similar to the one leading to Eq. (21), now yielding:

$$f(s) = \frac{\sqrt{1-\rho_{12}^2}}{\pi a_{12}} exp\left(\frac{\rho_{12}}{a_{12}} s\right) K_0\left(\frac{|s|}{a_{12}}\right), \qquad a_{12} \equiv (1-\rho_{12}^2)\sigma_1\sigma_2. \tag{32}$$

The next step is to write the PDF for $z = s_1 - s_2$ using Eq. (A3) as in Eq. (23); to make the integral tractable, we assume

$$\rho_{12} = -\rho_{34} \equiv \rho. \tag{33}$$

In this case, the exponentials in front of Eq. (32) cancel their "s" dependence when computing the product $f_1 f_2$ in Eq. (23), contributing just a factor $exp(\frac{\rho_{12}}{a_{12}}z)$, which drops out of the integral. The problem is thus reduced to the evaluation of the convolution of a function $K_0(\frac{|s|}{a_{12}})$ with itself, which yields an exponential as we saw in the previous section. The final result can be expressed as:

$$f(e_l) = \frac{1}{2\sigma_{v_i}\sigma_{v_j}} exp\left[-\frac{|e_l|}{\sigma_{v_i}\sigma_{v_j}(1 + sgn(z)\rho)}\right] \tag{34}$$

where $sgn(z)$ is the function "sign of z" and $\rho = \langle e_l \rangle/(2\sigma_{v_i}\sigma_{v_j})$. Interestingly, the PDF for e_l is an exponential $f \sim exp(-|e_l|/\sigma_+)$ when $e_l > 0$, and a different exponential $f \sim exp(-|e_l|/\sigma_-)$ when $e_l < 0$, with $\sigma_+/\sigma_- = (1+\rho)/(1-\rho)$. When $\rho = 0 = \langle e_l \rangle$, Eq. (34) reduces to the uncorrelated result Eq. (27).

We can compute the standard deviation of $z = e_l$ from Eq. (34). However, it is better to do it in terms of s_1 and s_2, which will give us a result for generic ρ_1 and $\rho 2$. Eq. (30) simply states $\langle s_1 \rangle = \rho_{12}\sigma_1^2\sigma_2^2$, and $\langle s_1^2 \rangle = (1 + 2\rho_{12}^2)\sigma_1^2\sigma_2^2$ by integration from Eq. (32). We now use these results (and the analogous forms for $\langle s_2 \rangle$ and $\langle s_2^2 \rangle$) to obtain:

$$\sigma_{e_l} = \sigma_z = \sigma_1\sigma_2\sqrt{2 + \rho_{12}^2 + \rho_{34}^2} \tag{35}$$

Note that Eq. (35) reduces to Eq. (14) when $\rho_{12} = 0 = \rho_{34}$.

C. Cross helicity-type correlated Gaussians

We focus now on a model for the case in which the cross helicity is finite (see the discussion in the previous section) and the mean induced electric field is zero (or $H_c \neq 0,\ \epsilon = 0$). In terms of the $\{x_i\}$ variables, we will have to consider correlations for the pairs (x_1, x_4) and (x_2, x_3). We apply Eq. (A3) in just one step:

$$f(z) = \int f_{14}(x_1, x_4) f_{23}(x_2, x_3) \delta[z - (x_1 x_2 - x_3 x_4)] dx_1 dx_2 dx_3 dx_4 \tag{36}$$

where both f_{14} and f_{23} are given by Eq. (29). It is trivial to verify that in this case there is no mean electric field (for instance, $\langle z \rangle = \langle x_1 x_2 \rangle - \langle x_3 x_4 \rangle = \langle x_1 \rangle \langle x_2 \rangle - \langle x_3 \rangle \langle x_4 \rangle = 0$). The standard deviation for the components of the electric field can be computed very easily as well:

$$\sigma_z = \sigma_1 \sigma_2 \sqrt{2(1 - \rho_{14} \rho_{23})} \tag{37}$$

After integration in Eq. (36) we obtain (See Appendix A for details):

$$f(z) = \frac{1}{\sqrt{2}\sigma_z} gexp\left(\rho_{14}, \rho_{23}, -\frac{\sqrt{2}}{\sigma_z}|z|\right) \tag{38}$$

where σ_z is given by Eq. (37) and we defined

$$gexp(\rho_1, \rho_2, z) \equiv \frac{\sqrt{1 - \rho_1 \rho_2}}{2\pi} \int_0^{2\pi} exp\left(\frac{\Theta_1}{\Theta_2} \sqrt{\frac{1 - \rho_1 \rho_2}{1 - \rho_1^2}}\ z\right) \frac{d\theta}{\Theta_1 \Theta_2} \tag{39}$$

$$\Theta_i \equiv \sqrt{1 - \rho_i sin(2\theta)} \tag{40}$$

Note that when $\rho_1 = \rho_2$, the exponential $exp(z)$ drops out of the integral in Eq. (39), giving the result:

$$gexp(\rho, \rho, z) \equiv e^z \tag{41}$$

It is evident from Eqs. (29) and (36) that

$$gexp(\rho_1, \rho_2, z) = gexp(\rho_2, \rho_1, z) \tag{42}$$

Figure 1 shows $gexp(\rho_1, \rho_2, -|z|)$, evaluated numerically for the cases $\rho_1 = \rho_2 = 0.9$ and $\rho_1 = -\rho_2 = 0.9$. The first case corresponds to an exponential, as indicated by Eq. (41), and hence has a Kurtosis of 6. The other case shows a flatter function, yielding a kurtosis of 8.89. In fact, when $\rho_1 \to 1$ and $\rho_2 \to -1$, then $z \to 2x_1 x_2$, which implies, according to Eq. (21), that $f(z) \to (\pi\sigma_z)^{-1} K_0(|z|/\sigma_z)$. That is,

$$\lim_{\rho \to 1} gexp(\rho, -\rho, -|z|) = \frac{\sqrt{2}}{\pi} K_0(|z|) \tag{43}$$

In particular, in this limit the Kurtosis is 9 (see Eq. (13)).

D. Variance anisotropic turbulence

So far we have assumed variance isotropy (i.e. Eq. (10)), since calculations are far more involved in the anisotropic case. Nonetheless, there is a limit that is analytically tractable. This is the limit in which

$$\lambda \equiv \frac{\sigma_3 \sigma_4}{\sigma_1 \sigma_2} \to 0. \tag{44}$$

Note that this limit would occur, for instance, if one of the fluctuating fields was plane polarized, but the other was variance isotropic; in this case, the components of $\mathbf{e}$ in the polarization plane would both satisfy Eq. (44). Let us write Eq. (A3) in the form:

$$f(z) = \int F\left(\frac{x_1}{\sigma_1}, \frac{x_2}{\sigma_2}, \frac{x_3}{\sigma_3}, \frac{x_4}{\sigma_4}\right) \delta[z - (x_1 x_2 - x_3 x_4)] dx_1 dx_2 dx_3 dx_4 \tag{45}$$

where $F(x_1, x_2, x_3, x_4)$ is the joint PDF for x_1, x_2, x_3, x_4. Making the change of variables $\tilde{x}_i \equiv x_i/\sigma_i$, $\tilde{z} \equiv z/(\sigma_1 \sigma_2)$, we can rewrite Eq. (45) as

$$f(\hat{z}) = \sigma_3 \sigma_4 \int F\left(\frac{\tilde{z} - \lambda \tilde{x}_3 \tilde{x}_4}{\tilde{x}_2}, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4\right) \frac{d\tilde{x}_2}{|\tilde{x}_2|} d\tilde{x}_3 d\tilde{x}_4, \tag{46}$$

and taking the limit $\lambda \to 0$,

$$f(\hat{z}) \to \sigma_3 \sigma_4 \int F\left(\frac{\tilde{z}}{\tilde{x}_2}, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4\right) \frac{d\tilde{x}_2}{|\tilde{x}_2|} d\tilde{x}_3 d\tilde{x}_4 \tag{47}$$

Note that the dependence in $(\tilde{x}_3, \tilde{x}_4)$ can be integrated in a first step, and finally the problem reduces to an integral in $\tilde{x}_2$. It is straightforward to show that both for uncorrelated Gaussians and cross helicity-type correlated Gaussians we obtain (see Eq. (21)):

$$f(e_l) \to \frac{1}{\pi \sigma_{v_i} \sigma_{v_j}} K_0\left(\frac{|e_l|}{\sigma_{v_i} \sigma_{v_j}}\right). \tag{48}$$

and for dynamo-type correlated Gaussians we obtain (see Eq. (32)):

$$f(e_l) \to \frac{\sqrt{1-\rho^2}}{\pi a} exp\left(\frac{\rho}{a} z\right) K_0\left(\frac{|z|}{a}\right), \qquad a \equiv (1-\rho^2)\sigma_{v_i}\sigma_{v_j}. \tag{49}$$

and $\rho = \langle e_l \rangle / \sigma_{v_i} \sigma_{v_j}$ Note that all the solutions in this limit are related to K_0 functions. We recall that K_0 diverges logarithmically at the origin, and decays faster than a simple exponential:

$$K_0(|z|) \approx -ln\left(\frac{|z|}{2}\right) + C, \qquad |z| \ll 1, \tag{50}$$

$$K_0(z) \approx \sqrt{\frac{\pi}{2}}\,\frac{e^{-|z|}}{\sqrt{|z|}}, \qquad |z| \gg 1, \tag{51}$$

$C \approx -0.577215$ is Euler's constant.

Fig. 2 shows the distribution function $f(z)$ for small but finite $\lambda = 0.1$. The Gaussian random variables $\{x_i\}$ for the plot are obtained with a Gaussian random number generator. Except for the discrepancy at the origin, the PDF for z is fairly close to the asymptotic solution Eq. (48). The plot shows that λ does not need to get too small for the PDF to be close to Eq. (48).

We note that even though variance isotropy implies $\sigma_{v_i}\sigma_{b_j} = \sigma_{v_j}\sigma_{b_i}$ (which we need to derive the results in the previous sections), such a relationship does not require variance isotropy. For instance, consider the situation of compressible MHD turbulence in presence of a DC field in the z direction. In this case, $\sigma_{v_x} \approx \sigma_{v_y} \gg \sigma_{v_z}$, and $\sigma_{b_x} \approx \sigma_{b_y} \gg \sigma_{b_z}$ [8]. We immediately see that the variance isotropic results are valid for e_z due to variance isotropy in the perpendicular planes (variance axisymmetry). For the perpendicular components of $\mathbf{e}$, on the other hand, the variance isotropic results are valid if $\sigma_{v_x}/\sigma_{v_z} = \sigma_{b_x}/\sigma_{b_z}$ and $\sigma_{v_y}/\sigma_{v_z} = \sigma_{b_y}/\sigma_{b_z}$, which might be approximately satisfied in some situations.

III. NUMERICAL RESULTS

We solve the standard MHD incompressible dissipative equations using a pseudo-spectral Fourier technique as described in [10]. We label our runs as follows: (i) *ISO run.* This is an isotropic run, with no DC field. We let $\mathbf{v}(\mathbf{x}$,t) and $\mathbf{b}(\mathbf{x}, t)$ relax in time, from an isotropic initial condition with no net mean magnetic field. (ii) *DC run.* We let a DC magnetic field $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$ act on the plasma, and study the relaxation of the MHD fields $\mathbf{v}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t) = \mathbf{B}_0 + \mathbf{b}(\mathbf{x}, t)$. Both runs start with isotropic, power-law, random-phase Gaussian ($\mathbf{v}$,$\mathbf{b}$) fields, with unit energy, zero cross helicity and zero magnetic helicity. More explicitly, the initial fields are such that

$$E^v(k) = E^b(k) = \frac{C}{1 + (k/k_{knee})^{5/3}}, \tag{52}$$

where $k_{knee} = 4k_0$, k_0 being the smallest wavenumber of the system. $E^v(k)$ and $E^b(k)$ are respectively the kinetic and the magnetic omnidirectional power spectra. We define $L_0 = 2\pi/k_{knee}$, our "energy containing scale", as the unit length, u_0 as the unit velocity and magnetic field (recall that the magnetic field is written in velocity units) and $t_0 = L_0/u_0$ as the unit time. The normalization constant C in Eq. (52) is chosen so that $\langle b^2 \rangle = \langle v^2 \rangle = 2\sum_k E^{v,b}(k) = 1$ (at t=0). For the DC run, $B_{dc} = 1$ as well. The magnetic Prandtl number is set to 1, the macroscopic Reynolds number is $R = u_0 L_0/\nu = 200$ and the resolution is 128^3 in all the runs. In all cases the solutions approximately satisfy variance isotropy: $\langle v_x^2 \rangle \approx \langle v_y^2 \rangle \approx \langle v_z^2 \rangle$, $\langle b_x^2 \rangle \approx \langle b_y^2 \rangle \approx \langle b_z^2 \rangle$, where the brackets $\langle\ \rangle$ mean here spatial average over the whole computational volume. The fluctuating fields $\mathbf{v}$ and $\mathbf{b}$ average to zero at all times: $\langle \mathbf{v} \rangle = 0 = \langle \mathbf{b} \rangle$.

Figure 3 shows the PDF for e_z for the ISO run, at $t = 3$. The PDF is remarkably close to the exponential function Eq. (27), as it turns out from the figure. The DC run yields a similar result, as shown in Fig. 4. The PDFs for the components of $\mathbf{v}$ and $\mathbf{b}$ (not shown) always lies close to a Gaussian, which is consistent with the quasi-normal assumption of this work, and in general with the quasi-normal character of turbulence.

A way to quantify the departure of the observed PDFs from the predicted values is to measure their Kurtosis'. For instance, the Kurtosis for any component of $\mathbf{v}$ and $\mathbf{b}$ is usually slightly above the Gaussian value of 3 (typical values are between 3 and 3.1 in our simulations). The Kurtosis for the components of $\mathbf{e}$ have an interesting behavior, shown in Figs. 4 and 5. It seems evident that the Kurtosis tends to be smaller than 6 in the ISO case, and greater than 6 in the DC case. Nonetheless, the departure from the predicted value of 6 is always relatively small.

IV. SUMMARY AND CONCLUSIONS

In the present paper we study the statistical properties of the induced turbulent electric field for a magnetohydrodynamic (MHD) plasma under the quasi-normal approximation. In Section II we assume exact Gaussianity for both the turbulent velocity field $\mathbf{v}$ and the turbulent magnetic field $\mathbf{b}$, and find analytical expressions for the PDF for the components of the fluctuating induced electrical field $\mathbf{e} = -\mathbf{v} \times \mathbf{b}$ in some cases of interest. The simplest and most appealing result is obtained when statistical independence is assumed for the components of $\mathbf{v}$ and $\mathbf{b}$. In this case, the PDF for e_l is simply an exponential, see Eq. (27). As mentioned above, statistical independence implies zero cross helicity H_c, and zero mean turbulent induced electric field $\langle \mathbf{e} \rangle$ (i.e., impossibility of a dynamo effect). In sections II B and II C we allow correlations between the components of $\mathbf{v}$ and $\mathbf{b}$, to extend the model respectively to the cases of finite $\langle \mathbf{e} \rangle$ and finite H_c. In the former case, we obtain an analytical result when $\langle v_i b_j \rangle = -\langle v_j b_i \rangle$, i.e. Eq. (34): the PDF turns out to be an exponential of the form $exp(-|e_l|/\sigma_+)$ for $e_l > 0$, and an exponential of the form $exp(-|e_l|/\sigma_-)$ for $e_l < 0$. The PDF for the latter case can be written in terms of a function $gexp(\rho_1, \rho_2, z)$, which also reduces to an exponential when PDF when $\langle v_i b_j \rangle = \langle v_j b_i \rangle$, as indicated by Eqs. (38-41).

All of these results assume variance isotropy for the turbulent fluctuations. In Section II D we show that the effects of variance anisotropy are measured by a single parameter $\lambda = \sigma_{v_j}\sigma_{b_i}/\sigma_{v_j}\sigma_{b_i}$ We find that for extreme variance anisotropy ($\lambda \to 0$), the three cases considered before (i.e. uncorrelated Gaussians, dynamo-type correlated Gaussians and cross helicity-type Gaussians) give solutions related to modified Bessel functions K_0 (see Eqs. (48) and (49)).

In Section III we report results from three dimensional numerical simulations of the MHD equations at moderate Reynolds numbers, where we investigate turbulent relaxation from a broad-band initial condition, with no cross helicity. The PDFs for the components of $\mathbf{e}$ are remarkably close to the exponential function predicted in Eq. (27), as shown in Figs. 3 and 4. The dynamical departure from the exponential PDF seems to depend on the presence of a background, DC magnetic field. Figs. 5 and 6 seem to indicate that in the absence of a DC field the Kurtosis is slightly smaller than 6, while in the presence of a DC field it is slightly greater than 6. More numerical exploration may be needed to confirm this trend. But in general, the comparison between the numerical and the analytical results allows us to be confident that the analytical results give a very good first order approximation to the problem, at least at moderate Reynolds numbers.

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APPENDIX A: SOME USEFUL RELATIONS

Let z be a random variable which depends on another random variable x, then

$$f(z) = \int p(z|x) f(x) dx \tag{A1}$$

where f is a PDF for its variable and $p(z|x)$ is the conditional probability distribution for z given x. Consider now the special case in which z is deterministically related to x by a known function $z = \psi(x)$. In this case, $p(z|x) = \delta(z - \psi(x))$:

$$f(z) = \int f(x) \delta(z - \psi(x)) dx \tag{A2}$$

The extension of this result to multiple dimensions is straightforward. If $\{x_i\}$ is a set of n random variables $\{x_1 ... x_n\}$, then

$$f(z) = \int f(x_1, ..., x_n) \delta(z - \psi(x_1, ..., x_n)) dx_1 ... dx_n \tag{A3}$$

where $f(x_1, ..., x_n)$ is the joint PDF of the $\{x_i\}$ variables.

We now show how to obtain the PDF for the cross helicity-type case, Eq. (38), from Eq. (36). Our first step is to make a change to non-dimensional variables, as in Eq. (46). We note that here $\lambda = 1$ (i.e., we are assuming variance isotropy). We replace $\delta(z - (x_1 x_2 - x_3 x_4)) = \delta(\sigma_1\sigma_2(\tilde{z} - (\tilde{x}_1\tilde{x}_2 - \tilde{x}_3\tilde{x}_4))) = (1/|\tilde{x}_1|\sigma_1\sigma_2)\delta(\tilde{x}_2 - \tilde{z}/\tilde{x}_1 - \tilde{x}_3\tilde{x}_4/\tilde{x}_1)$ in Eq. (36) and compute the trivial integral in $\tilde{x}_2$. We then compute the integral on $\tilde{x}_3$, which is of the form $\int exp(-A\tilde{x}_3 - B\tilde{x}_3^2)d\tilde{x}_3 = \sqrt{\pi/B}\ exp(A^2/4B)$, where both A and $B > 0$ are constant with respect to $\tilde{x}_3$. Rearranging terms and making a change to polar variables $(\tilde{x}_1, \tilde{x}_4) \rightarrow (r, \theta)$, we can write the remaining 2D integral as

$$f(\tilde{z}) = \frac{1}{(2\pi)^{3/2}\sigma_1\sigma_2\sqrt{1-\rho_{14}^2}} \int exp\left[-\frac{\Theta_{14}^2}{2(1-\rho_{14}^2)}r^2 - \frac{\tilde{z}^2}{2\Theta_{23}^2}r^{-2}\right]\frac{drd\theta}{\Theta_{23}} \tag{A4}$$

where we defined $\Theta_{ij} \equiv \sqrt{1 - \rho_{ij} sin(2\theta)}$. Making a last change in variables $r \rightarrow \hat{r}$ according to $\frac{\Theta_{14}^2}{2(1-\rho_{14}^2)}r^2 = |\tilde{z}|\hat{r}^2$, the integral in $\hat{r}$ takes the form $\int exp[-|\tilde{z}|(\hat{r}^2 - A^2/4\hat{r}^2)d\hat{r} \propto exp(A|\tilde{z}|)]$, where A depends on θ. The exact result is shown in Eq. (38).

REFERENCES

[1] H.K. Moffat, in *Magnetic field generation in electrically conducting fluids*, (Cambridge University Press, Cambridge, 1978).

[2] L.F.G. Simmons and C. Salter, *Proc. Roy. Soc. A* **165**, 73 (1938).

[3] D. Biskamp, in *Nonlinear Magnetohydrodynamics*, (Cambridge University Press, Cambridge, 1997), Chapter 7.

[4] Whang Y. C., during quiet conditions, *Solar Phys.*, *53*, 507 (1977)

[5] N. S. Padhye, C. W. Smith and W. H. Matthaeus, *J. Geophys. Res.* (2001) in press

[6] P.Y. Chou, *Chinese J. Phys.* **4**, 1 (1940).

[7] M. Millionshtchikov, *Dokl. Akad. Nauk. SSSR* **32**, 615 (1941).

[8] W.H. Matthaeus, S. Ghosh, S. Oughton and D.A. Roberts *J. Geophys. Res.* **101**, 7619 (1996).

[9] A.S. Monin and A.M. Yaglom, *Statistical Fluid Mechanics: Mechanics of Turbulence Volume I*, (MIT Press, Cambridge, MA, 1979), p. 231

[10] S. Oughton, E.R. Priest and W.H. Matthaeus, *J. Fluid Mech.* **280**, 95 (1994).

FIGURES

FIG. 1. The function $gexp(\rho_1, \rho_2, -|z|)$ for different parameters: $(\rho_1, \rho_2) = (0.9, 0.9)$ in solid line, and $(0.9, -.9)$ in dashed. The Kurtosis for these two cases is respectively 6, and 8.8

FIG. 2. In solid line, the PDF for a random variable $z \equiv x_1x_2 - x_3x_4$. The $\{x_i\}$ variables are generated with a Gaussian random number generator, and satisfy $\lambda \equiv \frac{\sigma_3\sigma_4}{\sigma_1\sigma_2} = 0.1$ (see Eq. (44)). In dashed line: the exact result for the PDF of z in the limit $\lambda \to 0$ (see Eq. (48)).

FIG. 3. In solid line, the PDF for one component of the electric field, as computed from the isotropic simulation, at t=3. In dashed line: the exponential PDF predicted by Eq. (27)

FIG. 4. In solid line, the PDF for one component of the electric field, as computed from the DC simulation, at t=3. In dashed line: the exponential PDF predicted by Eq. (27)

FIG. 5. A time series for the kurtosis of the components of the electric field. from the isotropic simulation. For each value of time, the values $K(e_x)$, $K(e_y)$ and $K(e_z)$ are used to compute a mean value (displayed as a cross) and the statistical error (displayed as a vertical bar).

FIG. 6. A time series for the kurtosis of the components of the electric field. from the DC simulation. For each value of time, the values $K(e_x)$, $K(e_y)$ and $K(e_z)$ are used to compute a mean value (displayed as a cross) and the statistical error (displayed as a vertical bar).

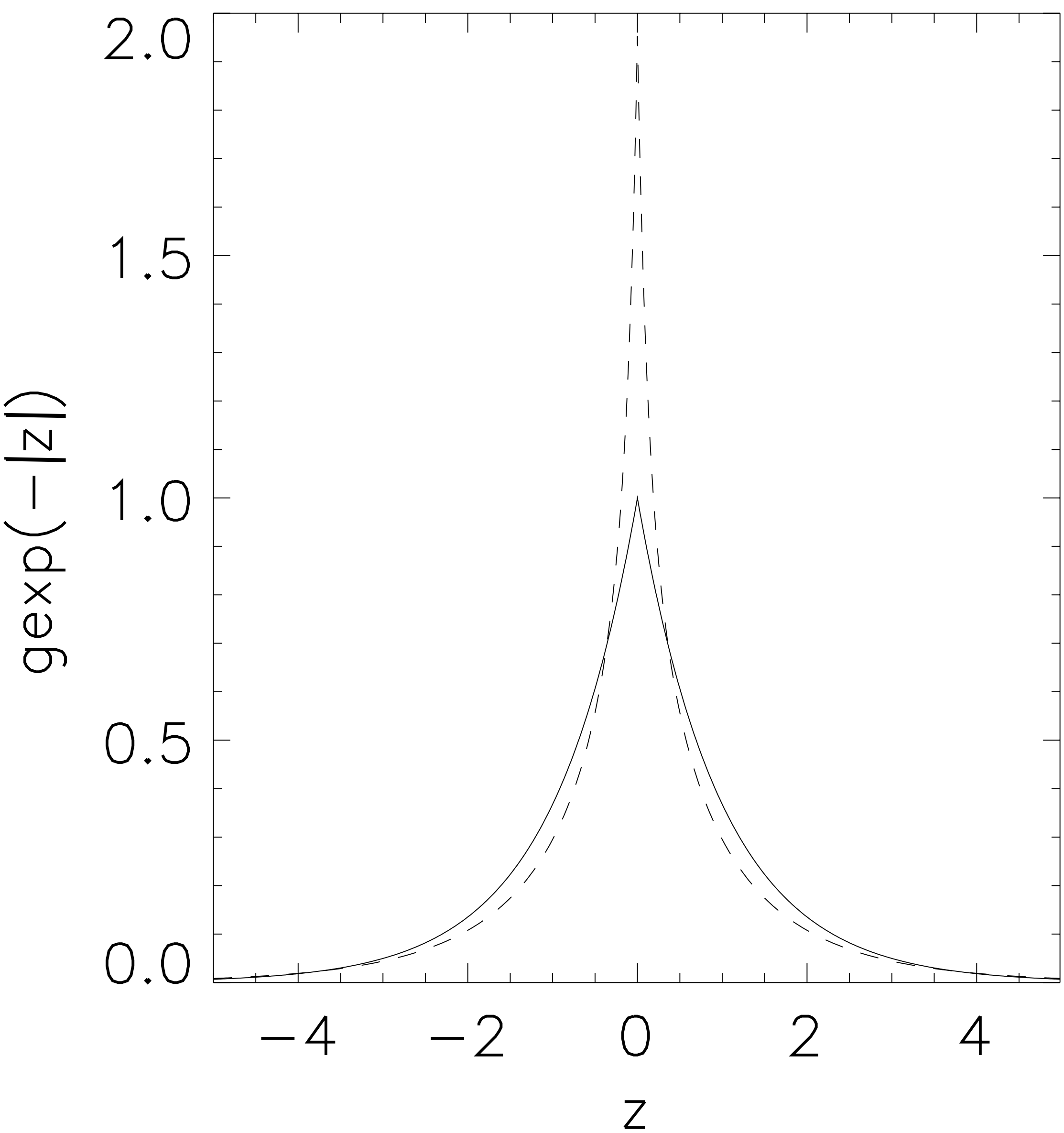

FIG 1.

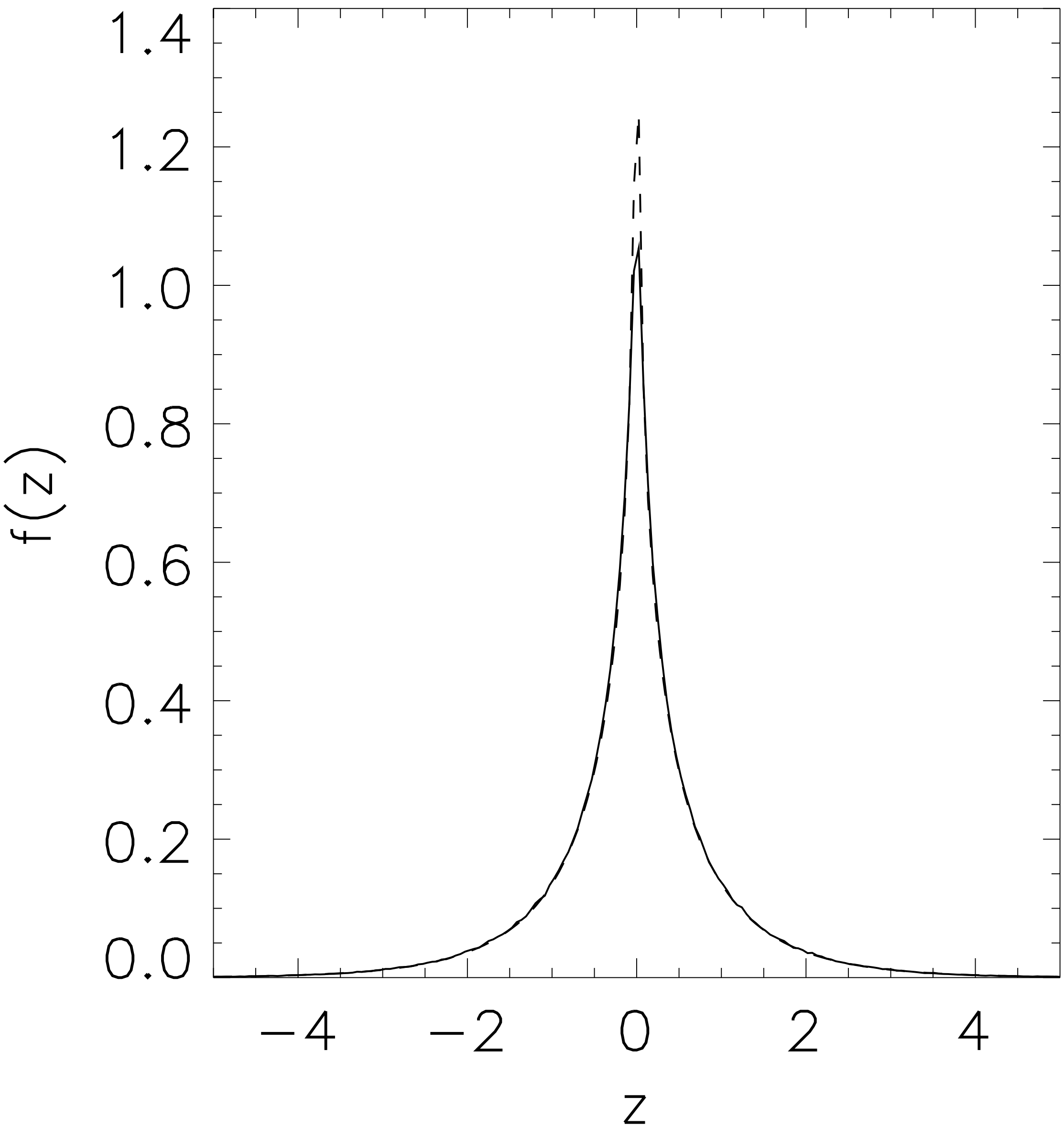

FIG 2.

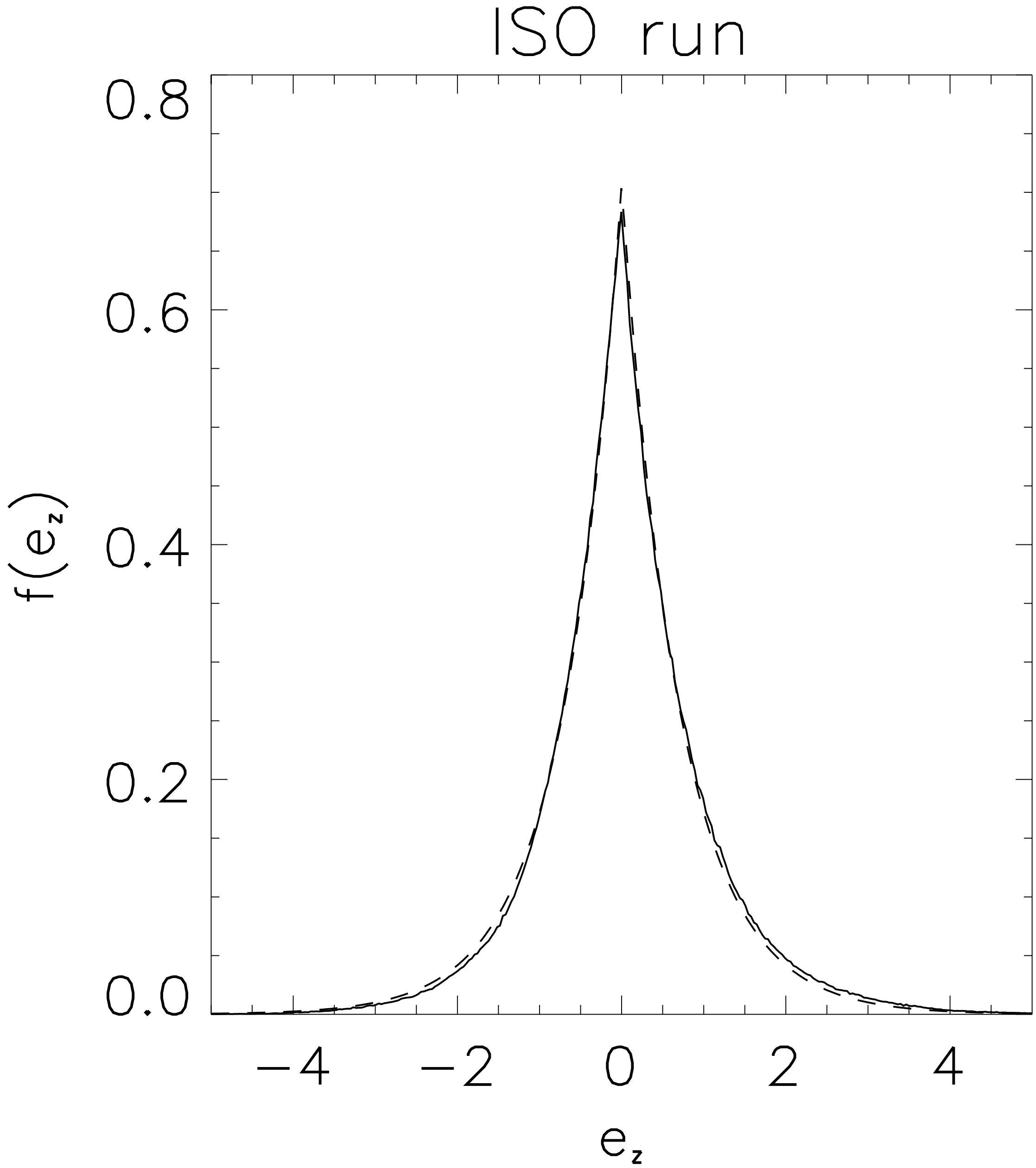

FIG 3.

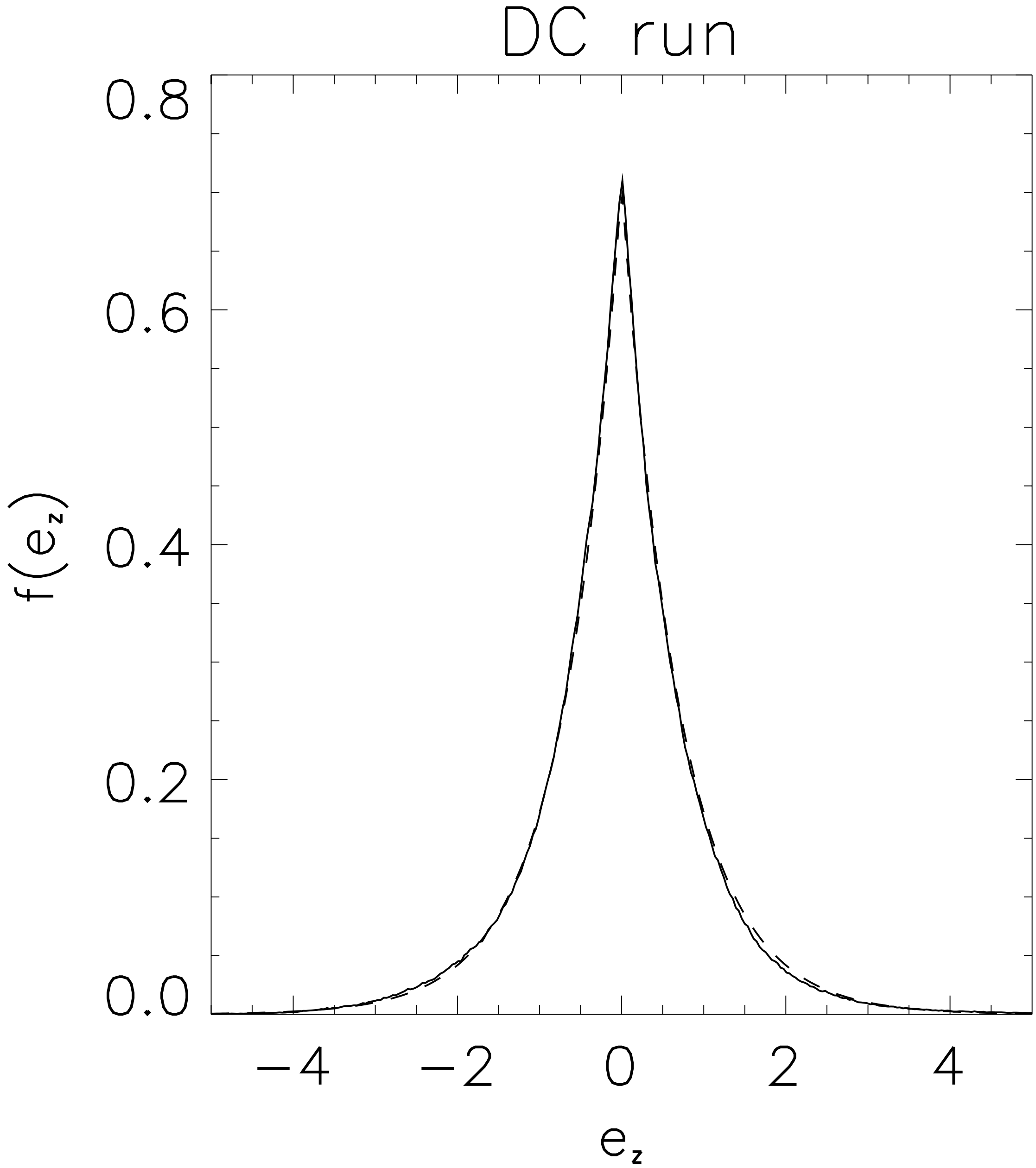

FIG 4.

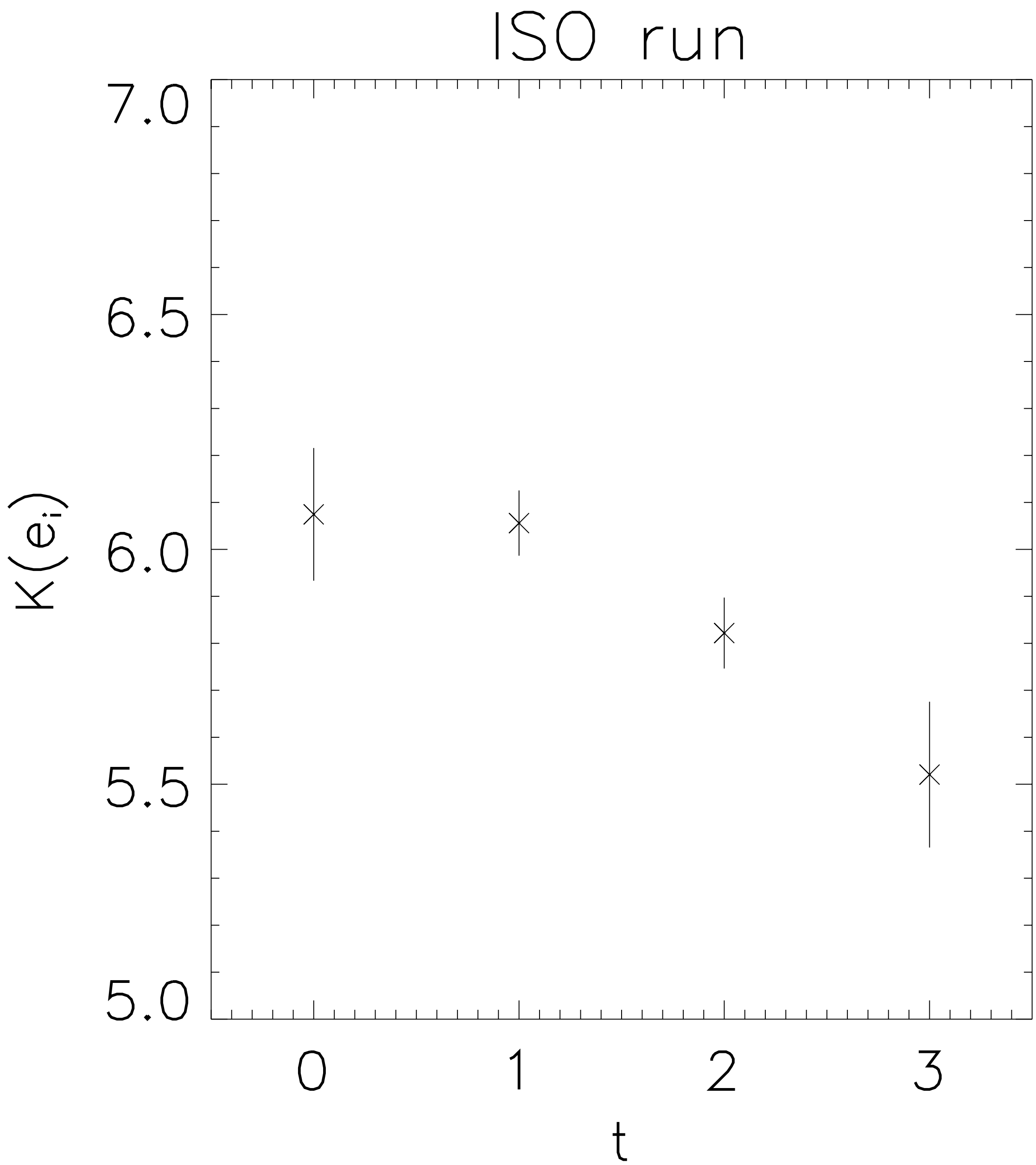

FIG 5.

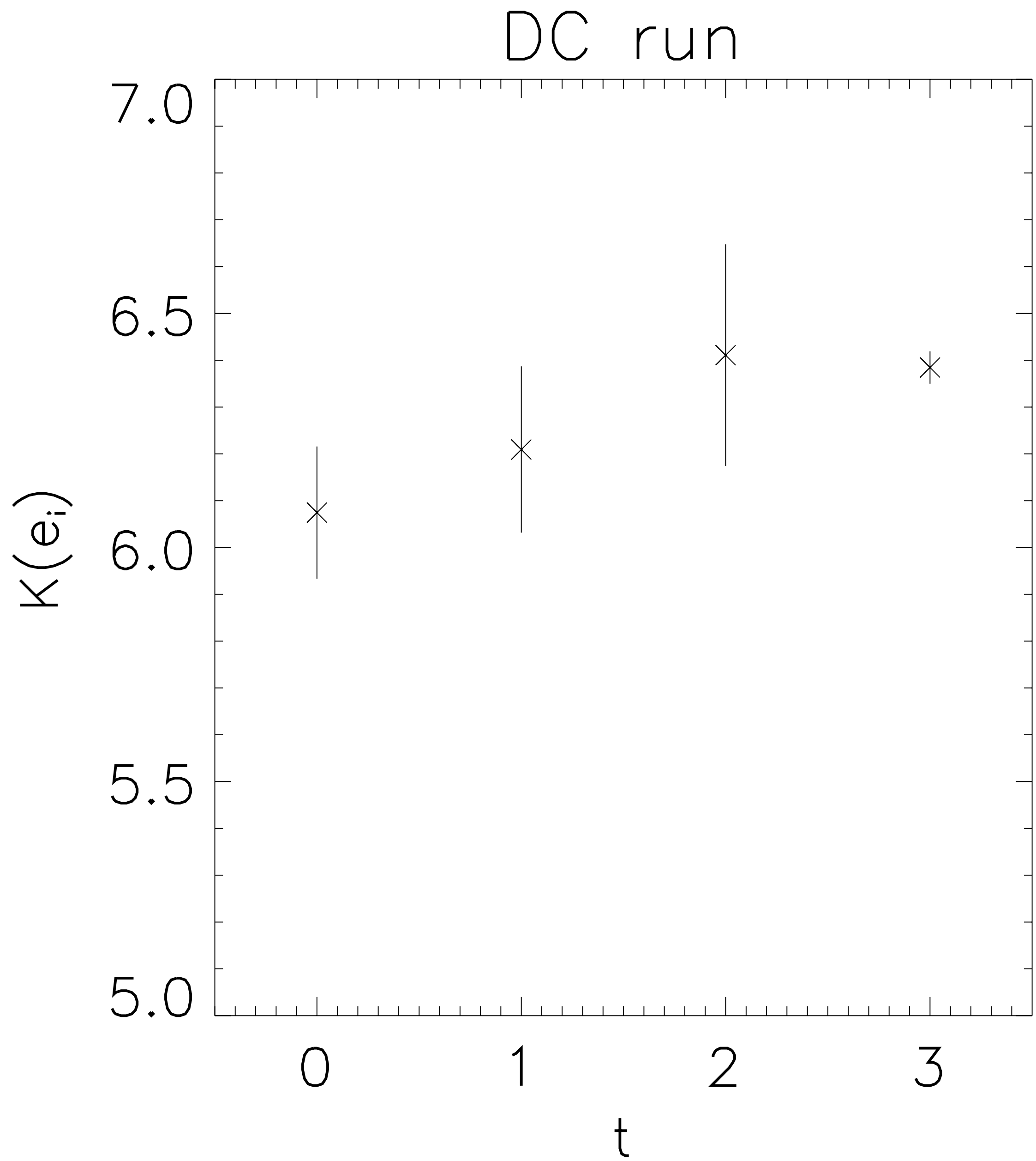

FIG 6.