V. Particle Transport and Acceleration

May-Britt Kallenrode, Chapter 6. Diffusive Transport

The remainder of this course will be concerned with energetic particle populations, their transport through background plasma and their acceleration. We have already encountered an example of such a particle population, the interstellar pickup ions. They are injected into the solar wind, start their motion perpendicular to the IMF (not necessarily along the solar wind flow direction!), and finally through scattering at waves they are assimilated into the solar wind frame. When looking more closely at their distribution it becomes clear that even in their final stage the pickup distribution only on average moves with the wind. Individual particles still move with respect to the solar wind plasma and thus are subject to transport effects. However, in this case we were not concerned with acceleration. The only energy the pickup ions **inherit** (they have this energy from the instant they are created!) and will be able to spend is their original kinetic energy in the rest frame of the plasma (in our case the solar wind), i.e. $E_{pickup} = \frac{1}{2} m_i v_{sw}^2$. That is why we studied their distribution in the solar wind frame of reference.

What happens then is that the ions are pitch-angle scattered to form a spherical shell in velocity space thereby perfectly conserving their original energy. Only in the rest frame of the observer (or the spacecraft) does it seem that some of them (the most energetic ones seen in the direction of the solar wind) have gained energies up to

$$E_{cutoff} = 4 \frac{1}{2} m_i v_{sw}^2 .$$

Apparently we have to add up energies in the transformation from the plasma rest frame into that of the observer. We see this by adding the respective velocities in performing the transformation:

$$\vec{u} = \vec{v} + \vec{v}_{sw}$$

So much for the purely mechanical picture, which seems to serve perfectly to understand the situation. However, we can also take an electrodynamic approach to uncover the electric fields hidden behind the scenery, which actually transfer the energy. It is the $\vec{E} = \vec{v}_{sw} \times \vec{B}$ electric field, which

- either directly (if the average IMF is sufficiently perpendicular to $\vec{v}_{sw}$) or
- slower indirectly (by means of magnetic fluctuation)

drive the pickup ions to move on average with the solar wind.

Before we attack the acceleration processes themselves at the hand of examples in space, let us look at the assumptions for the transport equations, which will be treated separately, and their consequences on the particle distributions.

V.1 Underlying Assumptions and Basics

Kirk, pp. 225-230

As we have noticed with the pickup ions and was stressed in Electro & Magneto Dynamics, charged particles cannot gain energy out of a magnetic field. There has to be an electric field to achieve this.

$$\frac{d\vec{p}}{dt} = e(\vec{E} + \vec{v} \times \vec{B}) \;=\; e(\vec{E} + \frac{\vec{p}}{m} \times \vec{B}) \qquad \text{Lorentz Force} \qquad \text{V.1_1}$$

(valid in the non-relativistic case)

$$\textit{If} \quad \vec{E} = 0 \qquad \vec{p} \bullet \frac{d\vec{p}}{dt} = 0 \qquad \text{because of the cross product.} \qquad \text{V.1_2}$$

Because we now deal with the microphysics we have to turn to an equation that involves phase space. The evolution of the particle distribution is controlled by

$$\frac{\partial f}{dt} + \frac{d\vec{r}}{dt} \bullet \nabla f + \frac{d\vec{p}}{dt} \nabla_p f = 0 \qquad \text{(Liouville Theorem)} \qquad \text{V.1_3}$$

The distribution function remains conserved along trajectories in phase space. Changes within the distribution manifest themselves in momentum changes $d\vec{p}/dt$, which are described in our case by the Lorentz Force. In principle, we have to add all the interactions between individual particles. However, in all our cases we can treat the interactions as solely caused by collective fields and thus use the Vlasow Equation. This means that we restrict the parameter space to collisionless plasmas, i.e. $\nu_{coll} << \nu_{plasma}$ (any collision frequency is much smaller than any relevant frequency in the plasma, i.e. plasma frequency, gyro frequencies etc.).

Because we treat a problem that involves plasma, the situation is complicated by the fact that the Lorentz Force itself depends explicitly on the position and the motion of the individual charged particles. The charge and current distributions can be described by Maxwell's Equations. Together this turns into a self-consistent system of equations. However, in most cases this is still too complicated. A split of the electric and magnetic fields into average fields (E_o, B_o) that affect the average distribution f_o and fluctuating fields (E_1, B_1), which are linearized as we did in the treatment of waves, is appropriate and actually used.

Going even one step further, in most of our cases the energetic particles can be treated as test particles in background plasma. We made a point that this is true for example for pickup ions in the inner heliosphere. These ions feel the waves that are present in the background plasma, which change their distribution. However, if energetic particles modify some of the plasma properties, it is only through electromagnetic fluctuations, hence a collective effect. Then the task is to search for the dominant wave mode that the energetic particles excite in the plasma. We make use of the linearized form of the wave description to deduce the effect of these waves on the particles.

Such a modification of the surrounding plasma applies, for example, to cosmic rays in the ISM. Let us look at this case more closely: Their energy density is: ≈ 1 eV/cm^3 is dominated by particles with 1-10 GeV. Their density is $n_{iR} = 10^{-10} - 10^{-9}$ cm^{-3}. The

parameters of the interstellar gas are T ≈ 7000 K ≈ 0.7 eV and n_{ISM} = 0.1 – 1000 cm^{-3}. Therefore, the energy density of the cosmic rays is comparable with that of the interstellar medium, while their number density is minute. Cosmic rays are an example where their presence matters to the background plasma. Waves and disturbances will be created that act again on cosmic rays. Ultimately both components are coupled.

Pickup ions are a perfect example, where we can study some of the same features, starting even from a simple test particle picture. In their case a stochastic process, i.e. pitch angle scattering, can immerse a small particle population almost perfectly into the rest frame of a background plasma, which has a significant bulk flow w.r.t. the rest frame in which the population originated.

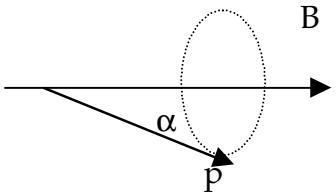
Let us look at some of the interactions more closely. The gyroradii of energetic particles are generally large compared with the length scales of the fluctuations (wavelengths) in the plasma. In the few cases (injection of particles into acceleration process out of the thermal plasma) where this is not the case and strong interaction with the plasma occurs, we have to be very careful and rethink our assumption. Generally, this means: long wavelengths and low frequencies are important, i.e. Alfvén waves. We can talk about small deviations of the trajectory in the individual "collision" with waves.

These Alfvén waves are fixed to the plasma frame. Energetic particles bounce off such fluctuations, leave the plasma rest frame intact (as if the mass of the collision partner - Alfvén wave - were very heavy) and therefore change only direction but not their energy! Furthermore: $V_A << v_{Bulk}$.

The distinction between these 2 basic quantities (energy or momentum and direction) is reemphasized by the choice of the coordinates for the momentum distribution (the velocity is equivalent in non-relativistic cases).

$f(p, \mu)$ where p is the total momentum and

$\mu = \cos \alpha$



the cosine of the pitch angle α.

We assume (and this is good for almost all our cases) gyrotropic distributions, thus f is constant in the 3rd coordinate. Because p remains primarily unchanged, the basic transport process is pitch angle scattering, and the change of f due to fluctuations occurs in μ only. It is a random walk that can be described as diffusion:

$$\left.\frac{df}{dt}\right|_{scatt.} = \frac{\partial}{\partial\mu}\left(D_{\mu\mu}\frac{\partial t}{\partial\mu}\right) \qquad \text{V.1_4}$$

and thus with the complete derivative:

$$\frac{\partial f}{\partial t} + \vec{v}\nabla f = \frac{\partial}{\partial\mu}\left(D_{\mu\mu}\frac{\partial f}{\partial\mu}\right) \qquad \text{V.1_5}$$

This looks similar to a description of spatial diffusion of particles

$\partial n/\partial t = \nabla(D\nabla n)$ [from $nv_{Diffusion} = -D\nabla n$ and $\partial n/\partial t + \nabla(nv_{Diffusion}) = 0$]

Ultimately, this leads to isotropization of the particle distribution in the plasma rest frame (Problem 3, HW 5), after the particles have been injected at initial pitch angle μ_o. The grip of the magnetic field keeps the energy of the particles basically constant. On time scales much longer than this isotropization time diffusion in p (or energy space) may become important, too. Only through the motion of the fluctuations (i.e. with their speed v_A) this grip can be softened, i.e. by collision with moving scattering centers energy can be exchanged, similar to elastic collisions with moving particles. Thus energy (momentum) diffusion can occur as described by

$$\left.\frac{df}{dt}\right|_{p\,scale} = \frac{1}{p^2}\frac{\partial}{\partial p}\left(D_{pp}\frac{\partial f}{\partial p}\right) \qquad \text{V.1_6}$$

Whereas $D_{\mu\mu} = v\lambda/3$ scales with the particle velocity v, $D_{pp} = V_A\lambda/3$ scales with V_A, and so typically:

$$D_{pp}/D_{\mu\mu} \approx V_A^2/v^2$$

V_A < 100 km/s (typical value) is generally small.

For example, for pickup ions we get:

$V_A \approx 50$ km/s in the solar wind and: $v = v_{sw} = 440$ km/sec

Therefore, $D_{pp}/D_{\mu\mu} \approx 1/100$, i.e. really small. This is why the cutoff of the pickup ion distribution is so sharp!

V.1.a Mechanical Picture of Diffusion

We mentioned the similarity of the description of scattering and diffusion in momentum space to that in "real" space. Let us use this analogy to see how diffusion is connected with collisions or scattering at electromagnetic fluctuations that can be treated accordingly.

Let us start with a simple situation for collisions in one dimension: By the tossing of a coin it is determined, whether a piece on the game board moves one unit to the left or to the right. Therefore, on average after a long time the distance Δx from the origin:

$\langle\Delta x\rangle = 0$ but $\langle\Delta x^2\rangle \neq 0$ This behaves like the mean quadratic error. If you don't know the game and its dimensions you can calculate the unit step or the average step size λ after N bounces:

$$\langle\Delta x_N^2\rangle / N = \lambda^2 \qquad \text{(V.1_7)}$$

This is the result of a statistical random walk in 1-dimension. For 3 dimensions we get

$$\sum_i \langle\Delta x_{iN}^2\rangle / N = 3\langle\Delta x_N^2\rangle / N = \lambda^2 \qquad \text{(V.1_8)}$$

because the particle has 3 degrees of freedom to move. A particle with speed v covers the distance $s = v\cdot t = N\lambda$ after N steps. Thus:

$$3\langle\Delta x_N^2\rangle = N\lambda^2 = \lambda v t \qquad \text{(V.1_9)}$$

The diffusion coefficient D is defined through the average squared distance $\langle\Delta x^2\rangle$ from the origin after time t:

$$D\cdot t = \langle\Delta x^2\rangle \text{ thus: } D = \lambda v/3 \text{ for 3-D diffusion} \qquad \text{(V.1_10)}$$

If we do this with many particles from the same origin, we will find a distribution in actual Δx around the origin that resembles a Gaussian. One can do this experimentally with a Galton board, having the particles enter at the same location all the time. In a homogenous gas the motion of all individual particles will have no net result, because they start all at different locations, and the outcome is another rendition of the homogenous gas.

However, if you release a tiny bit of perfume from a flask at a specific location, all people in the room can smell the perfume after some time. The perfume molecules will have diffused from their origin. They will form such a Gaussian distribution. Now there is a source, i.e. the flask, and thus a huge gradient in perfume density n. Because more collisions will send molecules away from the flask than towards it, this results in a net flux:

$$S = -D\nabla n \qquad \text{(V.1_11)}$$

We call S the streaming of particles caused by the diffusion. It is the net flux density from the random walk of particles in all directions. The gradient is the driving force, and the diffusion coefficient D (or tensor in the case of direction dependence) represents the mobility of particles. As can be seen, it depends on particle speed v and mean free path λ.

With the continuity equation, we can rewrite equation V.1_11: (V.1_12)

$$\frac{\partial N}{\partial t} + \oint_A S dA = 0$$

Here N is total number of particles released, S represents the streaming through the surface A. With the particle density n this turns into:

$$\text{or with Gauss' Theorem: } \partial n/\partial t + \nabla S = 0 \qquad \text{(V.1_13)}$$

With V.1_11 we get the diffusion equation: (V.1_14)

$$\frac{\partial n}{\partial t} = \nabla(D\nabla n) \quad \textit{or}: = D\nabla^2 n \quad \textit{if D independent of x}$$

Let us now look at a spherically symmetric problem with a source $G(r_o, t)$. Then:

$$\frac{\partial n(r,t)}{\partial t} - \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 D\frac{\partial n(r,t)}{\partial r}\right) = G(r_o,t) \qquad \text{(V.1_15)}$$

N_o particles are released at r = 0 at t = 0. The solution then reads:

$$n(r,t) = \frac{N_o}{\sqrt{(4\pi Dt)^3}}\exp\left(-\frac{r^2}{4Dt}\right) \qquad \text{(V.1_16)}$$

With the two competing dependencies on t this function quickly rises (exponentially) to a maximum and then decays (proportional to $t^{-3/2}$). This is the description of, for example, energetic particles released at a point source and then propagating by diffusion. The time of the arrival of the maximum is $t_{max}(r) = r^2/6D$. Using $D = \lambda v/3$, the time of maximum becomes $r/2\lambda$, i.e. is a measure of the number of scattering

"collisions" between the origin and the observer. Because the mean free path depends on the particle energy, mass and charge (we will make some more remarks on this later), one can use the observation of the time profile of arriving energetic particles as an indirect measure to determine their charge. Note: that we can more easily determine the energy and mass (i.e. species) of the particle.

Diffusion Convection Equation

If diffusion is described in a moving medium, i.e. for example the solar wind, we need to add a convective term to the equation. This means we include a convective streaming $S_{conv} = nv$, and the total streaming is $S = nv - D\nabla n$, i.e. diffusion can be countered by convection. The full equation reads:

$$\frac{\partial n}{\partial t} + v\nabla n = D\nabla^2 n \quad (\textit{for } D \textit{ independen of } r) \qquad \text{(V.1_17)}$$

The solution only requires replacement by r' = r-ut in the exponential term.

After completing this short excursion into regular spatial diffusion, we see, how pitch angle diffusion and momentum diffusion are related from the physics point of view. They describe a similar process in momentum and angle space. Scattering at waves in this case leads to a redistribution of particles in momentum space. Injection of a particle, for example, with pitch angle μ_o leads after some time to a distribution that is widely dispersed in pitch angle around μ_o. However, these processes are not only related in their description they are also causally related. Assume a particle gyrates in magnetic field with its fixed original pitch angle. We can easily follow the gyro center of this particle over time with a simple equation of motion. If we add pitch angle diffusion each particle will move differently along the field line. Therefore, the originally nicely peaked distribution in space will disperse along the field line. Through the motion in the magnetic field the pitch angle diffusion can be related to the mean free path in space by:

$$\lambda = 3/8\, v \int_{-1}^{1} \frac{\left(1-\mu^2\right)^2}{D(\mu)} d\mu \qquad \text{(V.1_18)}$$

V.1b Physical Picture of Scattering

After this more or less heuristic excursion into the mechanical picture of diffusion let us make a few remarks on the microphysics behind the scattering. As outlined above, the microphysics of the particles in the distribution function that controls scattering and thus diffusion goes ultimately back to Liouville's Theorem.

$$\frac{\partial f}{\partial t} + \nabla_x(\vec{v}f) + \nabla_p(\vec{F}f) = 0$$

with $\nabla_x(\vec{v}) = 0$ ***and*** $\nabla_p(\vec{F}) = 0$ ***we get***:

$$\frac{\partial f}{\partial t} + \vec{v}\nabla_x f + q\left[\vec{E} + \vec{v} \times \vec{B}\right]\nabla_p f = 0 \qquad \text{(V.1_19)}$$

Including Maxwell's Equations:

$$\nabla \times \vec{E} = \frac{1}{c}\frac{\partial \vec{B}}{\partial t} \quad \textbf{and}$$

$$\nabla \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + 4\pi\varepsilon_o \sum_s q_s \int f_s \vec{v} d^3\vec{p}$$

These are the Vlasow-Maxwell Equations for a "smoothed" distribution function f(**x**,**p**,t). We include fields self-consistently, i.e. determined by the distribution functions of all species s, but no collisions (with particles). We have a collisionless plasma.

1. We start from a spatially-homogenous equilibrium with **E** = 0 and **B** = $\mathbf{B_o}$. This leads to:

   $$q\left(\vec{v} \times \vec{B}_o\right) \bullet \nabla_p f = 0 \quad \textbf{and} \quad \frac{qB_o}{\gamma m}\frac{\partial f}{\partial \phi} = 0$$

   We can write: $f = f(p_\perp, p_\parallel)$; ***gyrotropic***

2. F > 0

3. Vlasow Equation is reversible.

We consider transport in a turbulent electromagnetic field **E, B** (leaving out any complexities of the fields). We only use average quantities, i.e. use ensemble averages.

$$\langle \mathbf{E} \rangle = 0; \langle \mathbf{B} \rangle = \mathbf{B_o}; \langle f \rangle = f_o(p_\perp, p_\parallel, t)$$

$$E = \delta E; \; B = B_o e_z + \delta B; \; f = f_o + \delta f$$

For the macroscopic fields the Vlasow Equation becomes simple and does not contain E or any reference to the distribution function any more. $\partial f_o/\partial t$ would be 0, if there were not the fluctuations of the turbulent fields. Therefore, we have to retain additional terms. We are looking for long term effects, i.e. the ensemble averages, as stated above. All strictly linear terms of averages over the fluctuations are 0. Thus we have to keep the next higher order, i.e. quadratic term:

$$\frac{\partial f_o}{\partial t} = -q\left\langle (\delta E + v \times \delta B) \bullet \frac{\partial \delta f}{\partial p} \right\rangle \qquad \text{(V.1_20)}$$

This acts as "effective collision term" and can be transformed into an effective "mean free path", thus leading to D etc.. However, we need to evaluate the fluctuation of f in first order.

$$\frac{\partial \delta f}{\partial t} + \vec{v}\nabla_x(\delta f) + q(v \times B_o)\nabla_p(\delta f) = -q(\delta E + v \times \delta B)\nabla_p f_o$$

We only need to consider the combination of the fluctuations in the fields with the smoothed distribution f_o. All other terms are 3rd order and thus to be neglected. This is called quasi-linear theory. What is now left, is to specify the wave spectrum that scatters the particles. In the solar wind ion cyclotron waves are effective scatterers. We find for these waves as resonance condition:

$\omega - kv_z + \Omega = 0$, where $\Omega = qB/m$ is the gyro frequency.

A simple interpretation of the resonance condition is that particles only feel the wave field, if they get hit at the correct phase all the time (like a person on the swing).

By doing the ensemble average we find that only particles that satisfy the resonance condition are substantially influenced by the wave field. Therefore, the scattering efficiency depends on the parallel velocity of the particles. For a turbulent wave field and a realistic distribution the influence is statistical. Thus it translates truly into a random scattering, and we can write the Vlasow Equation as:

$$\frac{\partial f_o}{\partial t} + \vec{v}\nabla_x f_o + q\left[\vec{v}\times\vec{B}_o\right]\nabla_p f_o = \nabla_p\left(\underline{\underline{D}}\bullet\nabla_p f_o\right) \qquad \text{(V.1_21)}$$

The second term describes convective change, the third drift in the homogeneous magnetic field, and the term on the right hand side describes diffusion in phase space. Because the distribution is gyrotropic, we can write $f_o(\mu,p)$.

As discussed above pitch angle scattering (i.e. in μ) is much more effective than scattering in momentum (or energy). Therefore, this diffusion in phase space will first isotropize the distribution in the rest frame of the plasma. The particle distribution, after effective scattering, will now be convected with the bulk flow. Let us now see some consequences of the assimilation of energetic particles into the plasma frame and then see how these diffusive transport processes play also an important role in particle acceleration, after evaluating which types of acceleration processes can occur.