I. The Solar Wind

I.1 Early Observations

Brandt, p. 103 - 110

Amazingly enough, the lowest energy particle population, the solar wind, was predicted and even sort of accepted, before it could be measured in-situ. This is a success story of theoretical studies together with some remote observational clues. All the historic observations of sun-Earth interactions were too vague to lead to the solar wind. What finally lead to its discovery were

- comets and
- theoretical scrutiny

In 1943, *Hofmeister* published a paper on ionic comet tails driven by a corpuscular radiation. *Biermann* published several papers on this subject in the 50's, which provided ample evidence for what was later called the solar wind. Sketch, View

Comets show 2 tails. Both point away from the Sun, but at slightly different angles. The wide bowed tail was interpreted as the dust tail, pushed away from the Sun by means of radiation pressure. However, there is a 2nd significantly more straight and flaring tail, the ion tail. According to spectroscopic observations it contains, for example, lots of CO^+. Sketch

In general it was laborious to transform the observations in the sky into the orbit, thus I don't want to do this here. As a simplification let's assume a comet whose orbital plane is perpendicular to the line-of-sight. Then we can compute the speed V_{Tail} with which the tail is dragged out directly from the comet's orbital motion V_{Perp} and the angle ε of the tail w.r.t. to the comet-sun line according to:

$$\mathrm{Tan}\,\varepsilon = V_{perp}/V_{Tail} \qquad \text{I.1\_1}$$

Average value from observations: $\mathrm{Tan}\,\varepsilon \approx 0.074$

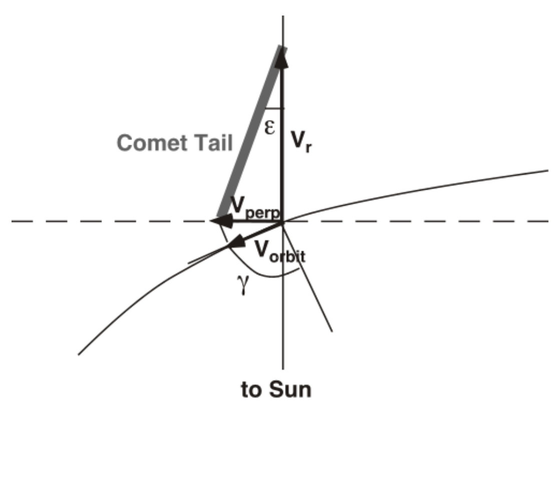
In reality the orbit is inclined w.r.t. the view in the sky. We see only a projection V_{perp} of the true orbital speed V_{orbit}., as shown in Fig. 1.

$$V_{perp} = V_{orbit} \cdot \mathrm{Sin}\,\gamma \approx 33.3 \text{ km/s}$$

from 600 comets $\rightarrow V_r \approx 450$ km/s

Therefore, a typical radial speed of 450 km/s for the ions in the tail emerged. A driving wind was assumed by *Biermann*. In the 50's the "what and how" of the driving 'wind' was not clear yet.

Fig.: 1: View of comet motion and of a comet tail in the sky.



I.2 Solar Atmosphere

Kallenrode, p. 147 – 149, **Brandt p.6-10**

Let us now turn to the solar atmosphere.

- Photosphere $T \approx 5500$ K
- Chromosphere $T \approx 10{,}000$ K
- Corona $T \approx 1 - 2\ 10^6$ K

The reason that the solar atmosphere is so extended is simply that it is so hot. The evidence for the high temperature is the presence of spectral lines from highly ionized ions, such as Fe with charge states above 20 etc. Such lines were first interpreted as new unknown elements (e.g. Coronium), until it was realized from calculations that these could be lines from highly ionized known elements. Taking away up to the innermost electrons from ions requires a very high energy in the interaction, up to several keV per collision. Therefore, the electrons in the environment, i.e. the corona, need to have energies that high. This argument leads to T ≈ several million K.

Here we have the strange case that the cooking pot is hotter than the stove underneath. This is a puzzle, which we will not treat in this course. This is a subject of the Solar MHD course. Let me just mention that the motion of the magnetic fields in the solar atmosphere is most likely the culprit. Magnetic energy is converted into kinetic energy and thus heat. I say 'most likely', because the details of the processes are still under debate. The processes are currently one of the main puzzles studied by the SOHO spacecraft and they form one of the main drivers to finally build a Solar Probe.

Static Atmosphere Model of the Corona

Let us first compare the solar atmosphere with the atmosphere of planets. The scale height of the sun’s atmosphere is huge. In contrast the scale height of the Earth's atmosphere reaches only a few km.

$$N(h) = N_o \exp(-h/H) \qquad \text{I.2\_1}$$

On Mt. Everest only ≈ 30% of the pressure at sea level is found. ($H_{Earth} \approx 8$ km)

$$p(h) = p_o \exp(-h/H) \qquad \text{I.2\_2}$$

where $H = kT/mg$ according to the kinetic gas theory

Sun: Photosphere: $H_{sun}(\text{Photosphere}) \approx 180$ km
$H_{Sun}(\text{Corona}) \approx 60{,}000$ km 10% of R_s

However, this is still much too small compared with picture of corona. Note: This model assumes that all the atoms are bound to the sun!

Static Atmosphere Model of the Corona

The outermost layer of the Earth's atmosphere is the **exosphere**, the part of the atmosphere that is not completely bound to it anymore. The fastest particles of a thermal distribution ($v > V_{escape}$) may escape, and thus the atmosphere reaches much farther into space. Let us now explore this option for the sun.

If there was a perfect vacuum, the particles would escape freely, but there is still a small chance for collisions.

Condition: $$1 \leq \int_{h_c}^{\infty} \sigma N(h)\, dh \qquad \text{I.2\_3}$$

where σ is the cross section for collisions.

-> $N(h_c) = 1/(\sigma h_c)$ -> $\lambda(h_c) \approx h_c$

In this model the mean free path for collisions equals the scale height. Above h_c the atmospheric gases can freely escape. For the sun this scale height is at ≈ 2 R_s.

The gas of temperature T is described by a Maxwellian

$$f(v)dv = 4\pi(m/2\pi kT)^{3/2} v^2 \exp[-mv^2/(2kT)]\, dv \qquad \text{I.2\_4}$$

All particles with $v > v_{esc}$ at 2 R_s will leave the sun from here.

v_{esc} (Gas) = $(2GM_s/r)^{1/2}$
v_{esc} ≈ 620 km/s for the sun.

Sketch

However, there is a modification: The solar atmosphere is a plasma. Electrons with $T_e = T_p$ tend to move much faster, i.e. would leave the sun faster. They build up an electric field as they separate from the protons (100 kV for a separation of 1m at coronal density $N \approx 10^{14}\ m^{-3}$)

Let's assume:

$$eE = m_p g_s \beta \qquad \text{I.2\_5}$$

i.e. partial balance between the electric field and the gravitational field of the sun. The fraction β of the gravitational field is balanced by electric forces of the escaping electrons.

Hydrostatic equilibrium (including Pressure):

$$m_p g_s - eE = (1-\beta) m_p g_s = kT_p dN_p/(N_p dh) = kT_p d(\log N_p)/dh \qquad \text{I.2\_6}$$
$$m_e g_s + eE \approx \beta m_p g_s = kT_e d(\log N_e)/dh \quad (m_e/m_p << 1)$$

T and N equal -> β = 0.5. -> $m_{eff} = m_p/2$

Thus a factor of 2 goes into the escape formula:

-> V_{esc}(Plasma) = $(GM_s/r)^{1/2}$
V_{esc}(Plasma) ≈ 435 km/s for the sun.

This seems remarkably close to the comet value for the 'wind', but is useless, because it doesn't work!!

Ions that leave the sun with V_{esc} will approach $V \to 0$ for $r \to \infty$

I.e. the **wind stops** and would be very slow at Earth !! What is wrong here?

The exospheric model really only works for a neutral gas. It is a test particle picture with no interaction between the particles. The corona is a plasma, which needs to be treated in a collective picture. The long range Coulomb force comes in. We have also neglected the fact that the electrons are capable of transporting heat very efficiently. The escape is no longer an adiabatic process.

Static Corona

***Kallenrode, p. 149– 152,* Kivelson&Russell, p. 91 – 100, Hundhausen p. 1-14**

The solar atmosphere is definitely bound in its lower layers. In spite of its high temperature the corona still seems bound.

$$E_{pot} = \frac{GM_s \; m_H}{R_s} \approx 2\text{keV}$$

$$E_{kin} = 160 \text{ eV} \quad \text{for } T = 2 \cdot 10^6 \text{K}$$

In 1957 Chapman tried a treatment of a static corona with thermal conduction. Heat conduction, which is not efficient enough inside the sun to compete with radiation, becomes an important role in the corona. The plasma is electrically conductive $\rightarrow$ heat conductive. After *Spitzer*:

$$\kappa = \kappa_0 \; T_e^{\,5/2} \qquad \text{I.2\_7}$$

coefficient of thermal conductivity	κ_0 weakly dependent on n and T
electric conductivity	$\sigma \approx ne/m\nu \sim T^{3/2}$ (Coulomb collisions)
thermal conductivity	$\kappa \approx nT/m\nu \sim T^{5/2}$

(similar to the Wiedemann-Franz law in metallic conduction ($\kappa \sim \sigma T$). The electrons in the metal are treated as an electron plasma)

For a material in thermal equilibrium, radiation will prevail over conduction for high temperatures since:

radiation transport	$\sim T^4$
conductive transport	$\sim T^{2.5}$

But in the thin (optically thin) solar atmosphere radiative transport is relatively inefficient.

We can use (I.2_7) in the corona. Then the heat flux is:

$$q = -\kappa \cdot \nabla T_e \qquad \text{(I.2\_8)}$$

For a static ($\partial/\partial t = 0$) corona with no heat sources and sinks the heat transfer reads

$$0 = -\nabla\vec{q} = \nabla(\kappa \cdot \nabla\, T_e)$$

in spherical polar coordinates:

$$\frac{1}{r^2}\ \frac{d}{dr}\left(r^2\, \kappa_o\, T_e^{5/2}\, \frac{dT_e}{dr}\right) = 0 \qquad \text{(I.2\_9)}$$

can be integrated

1. integration:

$$r^2\, \kappa_o\, T_e^{5/2}\, \frac{dT_e}{dr} = C$$

2. integration: Separation of variables:

$$T_e^{5/2}\, dT_e = \frac{C}{\kappa_o}\ \frac{dr}{r^2}$$

$$\frac{2}{7}\, T_e^{7/2} = -\ \frac{C}{\kappa_o}\ \frac{1}{r}$$

For $r \rightarrow \infty \qquad T \rightarrow 0$

C chosen such that

$$T_e = T_{eo} \cdot \left(\frac{r_o}{r}\right)^{2/7} \qquad \text{I.2\_10}$$

with the constants such that $\left| -\ \frac{7}{2}\ \frac{C}{\kappa_o} = r_o T_{eo}^{7/2} \right|$

For $T_e = 2 \cdot 10^6$K at the base of the corona

$r_o = 1.05\ R_s$ -> $T_e = 4 \cdot 10^5$K at the Earth

Assumption of a static atmosphere:

Hydrostatic equilibrium:

$$\frac{dP}{dr} = -\ \frac{GM_s\ \rho}{r^2} \qquad \text{(I.2\_11)}$$

The corona is a plasma with

$$\rho = n\,(m_e + m_p) \approx n \cdot m_p \qquad \text{(I.2\_12)}$$

(equal number of + and - charge), but both particles contribute to the pressure: (m will stand for m_p)

$$P = 2\, n\, kT \qquad = 2\rho\, kT/m \qquad \text{(I.2\_13)}$$

insert (I.2_13, 12 and 10) into (I.2_11):

$$\frac{d}{dr}\left(2\,\frac{\rho(r)}{m}\, k\, T_{eo}\left(\frac{r_o}{r}\right)^{2/7}\right) = -\frac{GM_s\ \rho(r)}{r^2}$$

With the multiplication rule of differentials:

$$\frac{1}{r^{2/7}}\,\frac{d\rho(r)}{dr} - 2/7\,\frac{\rho(r)}{r^{9/7}} = -\frac{GM_s\, m}{2kT_{eo}r_o^{2/7}}\cdot\frac{\rho(r)}{r^2}$$

$$\frac{d\rho(r)}{\rho(r)} = 2/7\,\frac{dr}{r} - \frac{GM_s\, m}{2kT_{eo}r_o^{2/7}}\,\frac{dr}{r^{12/7}}$$

$$\ln\rho(r) = 2/7\ln r + \frac{7}{5}\,\frac{GM_s\, m}{2kT_{eo}r_o}\,\frac{r_o^{5/7}}{r^{5/7}}$$

Integrate from $r_o \rightarrow r$, $\rho_o \rightarrow \rho$

$$\rho(r) = \rho_o\left(\frac{r}{r_o}\right)^{2/7}\exp\left\{\frac{7}{5}\,\frac{GM_s\ m}{2kT_{eo}r_o}\left[\left(\frac{r_o}{r}\right)^{5/7}-1\right]\right\} \qquad \text{(I.2\_14)}$$

$$P(r) = P_o\ \exp\left\{\frac{7}{5}\,\frac{GM_s\ m}{2kT_{eo}r_o}\left[\left(\frac{r_o}{r}\right)^{5/7}-1\right]\right\}$$

For $r \to \infty$ $\qquad \rho(r) \to \rho_o\,(r/r_o)^{2/7}$

$\rho \to \infty$

Wow!! This means an infinite density at infinity. This indicates major trouble in the physics.

In addition, with $P_o = 2n_o\, k\, T_{eo}$ and the use of I.2_14

$$P(r) \rightarrow P_o\cdot\ e^{-7/10\,\frac{E_{pot}}{E_{kin}}}\ \text{(corona)} \qquad P_o\cdot e^{-140/16} \approx P_o\cdot 0.00016$$

Recall values from the beginning of the lecture:

$\rightarrow\ P_\infty \approx 1.6\cdot 10^{-4}$ of the pressure at the sun

with $n_o \approx 10^{14}\ m^{-3}$ and $T \approx 2\cdot 10^6 K$

$$P_o = 2\cdot\ 10^{14}\cdot 1.36\cdot 10^{-23}\cdot\ 2\cdot 10^6 = 5.6\cdot 10^{-3} N/m^2$$

$$1.6\cdot 10^{-4} \quad\rightarrow\quad \approx 10^{-6} N/m^2$$

compared with 10^{-13} - 10^{-14} N/m^2 in the interstellar gas

Note: The density of the local interstellar medium (LISM) is $n \approx 10^5\ m^{-3}$ and its temperature $T \approx 7000$ K. Even, if we admit that there is still a substantial uncertainty ($n \approx 0.06 - 0.2 \cdot 10^6\ m^{-3}$ are debated) and we add pressure of a so far unknown interstellar magnetic field and that the ionized fraction needs to be counted twice (c.f. I.2_13), the total pressure does not reach anywhere near the extrapolated pressure of a static corona.

Parker: "Probably it is not possible for the solar corona or indeed perhaps the atmosphere of any star to be in complete hydrostatic equilibrium out to large distances!"

I. 3 Expansion of the Corona (Solar Wind)

Kallenrode, p. 152 - 156
Kivelson&Russell, p. 100 - 103
Hundhausen p. 1- 14
Kirk et al., p 48-52
Isenberg p. 15 - 20

Since the cometary observations (*Biermann*) suggested a rapid outflow from the solar corona, *Parker* considered this possibility in his model:
For a constant outflow (I.3_5) has to be replaced by:

I.) $$\rho u \frac{du}{dr} = -\frac{dP}{dr} - \rho\frac{GM_s}{r^2} \quad \text{or} \quad u\frac{du}{dr} = -\frac{dP}{\rho dr} - \frac{GM_s}{r^2} \qquad \text{I.3\_1}$$

and becomes a **momentum transport** equation

Viewgraph momentum transport

In addition, the mass in the flow must be conserved, **Mass conservation**:

II.) $$\nabla(\rho \cdot u) = \frac{1}{r^2}\frac{d}{dr}(r^2\rho u) = 0 \quad \text{or} \quad r^2\rho\frac{du}{dr} + r^2 u\frac{d\rho}{dr} + 2r\rho u = 0 \qquad \text{I.3\_2}$$

Integrating yields:

$$4\pi r^2 nu = I = \text{constant total flux} \quad (\rho = mn)$$

We have
3 u, ρ, P, total of 5 variables (or 3 variables for a spherically symmetric case)
but only
3 equations (I.3_1), 1 equation (I.3_2)
In addition, we need the information about the energy transport. To simplify the treatment this is replaced by an **equation of state** for ρ and P.
Parker used a polytropic law

III.) $P = P_o\,(\rho/\rho_o)^\alpha$ or differential: $d(P/\rho^\alpha)/dr = 0$ with the polytropic index α. I.3_3

Eqns. **I.)** and **III.)** can be combined to form a complete differential by using

$$1/\rho^{\alpha}\, dP/dr - \alpha P/\rho^{\alpha+1}\, d\rho/dr = 0 \quad ->$$

$$\frac{1}{\rho}\frac{dP}{dr} = \frac{\alpha}{\alpha\text{-}1}\frac{d}{dr}\left[\rho^{\alpha-1}\frac{P}{\rho^{\alpha}}\right] = \frac{d}{dr}\left[\frac{\alpha}{\alpha\text{-}1}\frac{P}{\rho}\right]$$

$$-> \quad \frac{u^2}{2} + \frac{\alpha}{\alpha\text{-}1}\frac{P}{\rho} - \frac{GM_s}{r} = E \qquad \text{I.3\_4}$$

This is the total energy, which is a constant of the motion under these assumptions.

a) **Adiabatic case:**

no energy transport α adiabatic index (e.g. 5/3)

b) **Isothermal case**:

very efficient energy transport

T = constant α = 1

(used by *Parker*: in his first solution for the solar wind)

The latter case requires a modification of the energy equation:

$$\frac{u^2}{2} + 2\frac{kT}{m_p}\ln\rho - \frac{GM_s}{r} = E \qquad \text{I.3\_5}$$

The first term is the kinetic energy of the bulk flow, the third the gravitational energy. Without the thermal term this would just be a simple energy equation for a single particle motion in a gravitational field. With the thermal energy there is another reservoir to tap for the bulk flow. As can be easily seen, the isothermal case (T = const.) leads to a transfer of thermal energy into the bulk flow during expansion. In other words, the density decreases on the way. In the adiabatic case the thermal term will remain constant, and we are back to the simple relation between u and GM_S/r.

With P = 2 n kT and (I.3_1)

$$u\frac{du}{dr} = -\frac{2kT}{m}\frac{d\rho}{\rho dr} - \frac{GM_s}{r^2} \qquad \text{(I.3\_1a)}$$

$$-\frac{d\rho}{\rho dr} = \frac{du}{udr} + \frac{2}{r} \qquad \text{(I.3\_2a)}$$

Inserting (2a) into (1a) to eliminate ρ:

$$u\frac{du}{dr} = \frac{2kT}{m}\left(\frac{1}{u}\frac{du}{dr} + \frac{2}{r}\right) - \frac{GM_s}{r^2}$$

$$\frac{du}{dr}\left(u - \frac{2kT}{m\,u}\right) = \frac{4kT}{m\,r} - \frac{GM_s}{r^2}$$

or with $\frac{du}{dr} = \frac{1}{2u}\cdot\frac{du^2}{dr}$

$$\frac{1}{2}\frac{du^2}{dr}\left(1 - \frac{2kT}{mu^2}\right) = \left(\frac{4kT}{m\,r} - \frac{GM_s}{r^2}\right) \qquad \text{(I.3\_6)}$$

With the sound speed $c_s = \sqrt{\frac{2kT}{\alpha m}}$ this becomes

$$\frac{1}{2}\frac{du^2}{dr}\left(1 - \frac{c_s^2}{u^2}\right) = \left(\frac{2c_s^2}{r} - \frac{GM_s}{r^2}\right) \qquad \text{(I.3\_7)}$$

The right hand side switches signs. At least somewhere below a certain radius

$$R_S < r < r_c = \frac{GM_s}{2c_s^2}$$

$$T < \frac{GM_s\,m}{4kr_o}$$

where $r_o \approx R_S$ is the base of the corona.

At r_c the right hand side is = 0, i.e., either:

$$\frac{du^2}{dr} = 0 \qquad \text{at} \quad r = r_c$$

or

$$c_s^2 = u^2 \quad \text{at} \quad r = r_c \qquad \text{(I.3\_8)}$$

(8a) either maximum or minimum at r_c **Viewgraph (Topology)**
(8b) or monotonically increasing or decreasing (passing through critical point)

-> 4 classes of solutions
-> 2 reach the sound speed at r_c

Solution:

$$\left(\frac{u}{c_s}\right)^2 - \ln\left(\frac{u}{c_s}\right)^2 = 4\ln\left(\frac{r}{r_c}\right) + 4\frac{r_c}{r} + C \qquad \text{I.3\_9}$$

where C = -3.

Class 3 and 4 start with $u >> c_s \rightarrow$ unrealistic since $E_{pot} > E_{min}$

Class 1: solar breeze, always subsonic, which means:

$$\left(\frac{u}{c_s}\right)^2 < 1 \quad \left(\frac{u}{c_s}\right)^2 << \ln\frac{u}{c_s} \qquad \text{for } r\rightarrow\infty$$

$$\text{lu}\,\frac{u}{c_s} \approx -2\ln\frac{r}{r_c}$$

$$\frac{u}{c_s} \approx \left(\frac{r_c}{r}\right)^2 \qquad \text{(I.3\_10)}$$

Because of mass conservation $u\rho r^2 = \text{const}$ -> $\rho = \text{const}$
-> finite pressure $\rho kT/m$ as $r \to \infty$, which we excluded earlier!

Only class 2 realistic, i.e., transition from subsonic to supersonic flow!

An analogy is the deLaval nozzle (homework problem)
for $u < c_S$ the width of the width of the flux tube is decreasing
for $u > c_S$ it is increasing
-> to produce supersonic flow

In the solar wind the boundary conditions are such that the same thing happens. We have basically an expanding nozzle, but due to the heat conduction the sonic velocity is reached at a certain height.

Some estimates:

Critical radius: (potential energy at R_S = 2000 eV)

With $r_C = GM_S/2\,c_S^2$ and $c_S^2 = 2kT/m$

$r_C/R_S = GM_S m/R_S\ /4kT = 2000/4\cdot170$ ≈ 3 for $T = 2\cdot10^6$ K
≈ 6 for $T = 1\cdot10^6$ K

The critical radius is closer to the solar surface for a hotter corona and farther away for a cooler corona. I.e. a stronger wind with higher density is produced in a hotter corona. The important processes for the wind formation are confined to the immediate vicinity of the sun.

The spacecraft that came closest to the sun up to date were the 2 Helios probes in the 1970's. They reached 0.3 AU ($\approx 60\ R_S$). This already required a good thermal design of the spacecraft. To measure the wind formation in-situ will require a Solar Probe, which is anticipated to reach $\approx 4\ R_S$. The Solar Probe will definitely get into the acceleration region and will probably pass the critical radius in the equatorial region of the sun.
Viewgraph (Solar Probe)

Velocities:

$u_{SW} = 490 - 745$ km/s **Viewgraph (Parker's Sol.)**

Densities:

$N \approx 0.35 - 40\ \mathrm{cm}^{-3}$ for $1 - 2\ 10^6$ K

The densities in the SW are $n \approx 1 - 30\ \mathrm{cm}^{-3}$ i.e., the estimate seems reasonable. However, the trend in the estimate (high densities for high corona temperatures, i.e. high solar wind speeds) is opposite to the observations. Something is missing in our simple picture!

Variation of the Temperature with Distance

We have made drastic simplifications. Let us get some additional processes back in. Let us also use this to evaluate the physical system in the neighborhood of the critical solution:

Using $u \frac{du}{dr} = -\frac{dP}{\rho dr} - \frac{GM_s}{r^2}$ with $P = \rho c_s^2$ (I.3_11)

where the sound speed is also r-dependent. However, we consider only a small deviation from the isothermal behavior, i.e., $\alpha - 1 << 1$ so that c_s is still close to the actual sound speed. With the same manipulations as in I.3_6 we get a similar equation as I.3_7 with an additional term from the explicit r-dependence c_s^2 in P:

$$\frac{1}{2}\frac{du^2}{dr}\left(1 - \frac{c_s^2}{u^2}\right) = -\frac{dc_s^2}{dr} + \left(\frac{2c_s^2}{r} - \frac{GM_s}{r^2}\right) \qquad \text{(I.3\_12)}$$

simplified to:

$$\frac{1}{2}\frac{du^2}{dr}\left(1 - \frac{c_s^2}{u^2}\right) = -r^2\frac{d}{dr}\left(\frac{c_s^2}{r^2}\right) - \frac{GM_s}{r^2}$$

where c_s is only varying slowly with r. We use the Ansatz $c_s^2(r) \approx \left(\frac{R_s}{r}\right)^\eta c_{so}^2$ with $0 < \eta < 1$. $\eta = 0$ is the isothermal case.

$$\frac{1}{2}\frac{du^2}{dr}\left(1 - \frac{c_s^2}{u^2}\right) = (2+\eta)\frac{c_{so}^2}{r}\left(\frac{R_s}{r}\right)^\eta - \frac{GM_s}{r^2} \qquad \text{(I.3\_13)}$$

We evaluate the equation at the critical point. For $\eta << 1$ we can use a Taylor expansion in η for the first term on the r.h.s.. In this way we can determine r_c close to the isothermal case by setting the r.h.s. to 0:

$$\frac{2c_{so}^2}{r} + \frac{c_{so}^2}{r}\left[\eta + 2\ln\left(\frac{R_s}{r}\right)\eta\right] - \frac{GM_s}{r^2} = 0$$

Using r_{co} (the isothermal critical radius) in the correction term

$$r_c \approx \frac{GM_s}{2c_{so}^2}\left[1 - \left(\ln\left(\frac{R_s}{r_{co}}\right) + \frac{1}{2}\right)\eta\right]$$

In the isothermal case $r_c/R_s \geq 3$, i.e. certainly $r_c/R_s > e$.

$$r_c \approx \frac{GM_s}{2c_{so}^2}\left[1 + \left(\ln\left(\frac{r_{co}}{R_s}\right) - \frac{1}{2}\right)\eta\right] \qquad \text{(I.3\_14)}$$

Therefore, the correction term in the bracket is positive!
The critical radius increases for $\eta > 0$, i.e. when deviating from the isothermal case to a situation with less heat transport.

Viewgraph (Topology)

More importantly this allows a qualitative stability discussion of the critical solution. Assume the wind is on the critical solution for a certain η_o. Now there is a small perturbation away from the critical solution

a) to the right: There is not enough heat conduction, i.e. η is larger than η_o, we reach the breeze solution. However, because η is larger now, the critical point moves further out, catching up with the flow.

b) to the left: There is too much heat conduction, i.e. η is smaller than η_o, the flow is driven too hard and reaches the sound speed too early. However because η is smaller now, the critical point moves further in towards the actual the flow.

It should be noted that also the sound speed on the left hand side is affected by the variation of η. We follow a similar chain of arguments to arrive at similar conclusions.

In any case, the deviation from the critical solution is always corrected towards the critical solution, i.e. it shows a similar behavior as a stable equilibrium.

Treatment with explicit heat transport

The modification of the equation of state does not remedy the problem with the densities, noted earlier. Probably something else turned wrong through our simplifying assumptions.

Instead of an Equation of State let us now use an explicit
Energy transport equation:

III.) $$\frac{1}{r^2}\frac{d}{dr}\left\{r^2\left[\left(\frac{\rho}{2}u^2 + \frac{3}{2}P\right)u + Pu - \kappa\frac{dT}{dr}\right]\right\} + \rho u\frac{GM_s}{r^2} = 0 \qquad \text{I.3\_15}$$

Under the Differential: transport of energy with the flow
Remaining Term: change of energy in the gravitational field (may be any field)
1. term: kinetic energy of bulk flow
2. term: inner (thermal) energy
3. term: work of expansion/contraction
4. term: heat flux through thermal conduction

In addition:

$$\rho u r^2 \frac{GM_s}{r^2} = -\rho u r^2 \frac{d}{dr}\left[\frac{GM_s}{r}\right] = -\frac{d}{dr}\left[\rho u r^2 \frac{GM_s}{r}\right]$$

because of mass conservation ($d\rho u r^2/dr = 0$). This transforms I.3_15 into a complete Differential.
Therefore:

$$4\pi r^2\left[\left(\frac{\rho}{2}u^2 + \frac{5}{2}P - \rho\frac{GM_s}{r}\right)u - \kappa\frac{dT}{dr}\right] = \text{Total Energy Flux}$$

Now we can evaluate the processes close to the sun ($r < r_c$) in more detail:

$$\rho\, u^2/2 << 5/2P << \rho\, GM_s/r$$

subsonic flow bound atmosphere

-> maybe the outflow at low altitudes was wrong. We still have a bound atmosphere below the critical point, as pointed out earlier! This leaves only 2 significant terms at low altitude: the **gravitational term** and **heat conduction**. Possibly the heat flux is able to "lift" the atmosphere and therefore produce higher densities to start the flow.

With $\kappa = \kappa_o T^{5/2}$ and the **total energy flux** J:

$$-\kappa_o\, T^{5/2}\frac{dT}{dr} = \frac{J}{4\pi r^2} + \rho u\frac{GM_s}{r}$$

Separation of variables:

$$\kappa_o\, T^{5/2} dT = -dr\left[\frac{J}{4\pi r^2} + \rho u r^2\frac{GM_s}{r^3}\right]$$

$$T^{7/2} = \frac{7}{2\kappa_o}\left[\frac{J}{4\pi r} + \rho_s u_s R_s^2\frac{GM_s}{2r^2}\right]$$

$$T = \left[\frac{7}{2\kappa_o}\frac{J}{4\pi R_s}\frac{R_s}{r}\left(1 + \frac{4\pi\rho_s u_s R_s^2}{J}\frac{GM_s}{2r}\right]^{2/7}$$

$$T = \left[\frac{7}{2\kappa_o}\frac{J}{4\pi R_s}\frac{R_s}{r}\left(1 + \frac{r_2}{r}\right)\right]^{2/7}$$

where $\frac{2r_2}{R_s} = 4\pi\rho_s u_s R_s^2\frac{GM_s}{R_s}/J$

the ratio of the gravitational and net energy flux J at the solar surface. To make sense the net energy flux has to be at least smaller than half the gravitational flux. (**Why?**) Otherwise $r_2 < R_s$, i.e. below the surface!
We get:

$r < r_2$	$T \approx r^{-4/7}$	steep:	doing work lifting the gas
$r > r_2$	$T \approx r^{-2/7}$	flat:	not much work (temperature

dependence like stationary atmosphere)

1) r-dependence indicates: the situation is indeed far away from isothermal case!
2) $J > 0$ -> heat conduction must be enough to lift atmosphere up.

$$J = 4\pi R_s\kappa_o\, 2/7\, T^{7/2} - 4\pi\rho_s u_s R_s^2\frac{GM_s}{2R_s} > 0$$

Therefore, the density of the lifted atmosphere ρ_s is limited.